Effects of Wire Length in Turbulence Investigations with a Hot-Wire Anemometer*

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Summary: The paper discusses the influence of wire length on the characteristics of a turbulent flow as measured with a hot-wire anemometer. Most of the mathematical treatment can be directly applied to other problems of length correction, such as may occur, for instance, in problems of astrophysical turbulence, as well as to some problems related to measurements of random processes. The discussion will, however, use the example of a hot-wire anemometer with particular attention to the influence of the length of the wire on the measurement of the intensity of turbulence, correlation coefficients, and scale and microscale of turbulence.

Notation

The use of a bar over a symbol denotes a time average taken over a time interval of at least $2T$ as defined in equations (8) and (9).

Subscript "b" is used to denote the values determined by means of a wire of non-negligible length.

Subscripts A and B refer to the wires A and B (*see* Fig. 3).

The prime notation denotes a turbulent component and is also used in Section 4 to designate the derivatives (such as R', R'').

O, X, Y, Z	rectangular co-ordinate axes (*see* Fig. 2)
$\mathbf{V}(X, Y, Z, t)$	instantaneous vectorial velocity at time t
$\mathbf{V}'(X, Y, Z, t)$	turbulent vectorial velocity at time t
O, x, y, z	rectangular co-ordinate axes with Ox parallel to $\overline{V}$ (*see* Fig. 2)
u, v, w	instantaneous velocity components in the directions Ox, Oy, Oz
u', v', w'	turbulent velocity components in the directions Ox, Oy, Oz
$d, i, R, R_a, R_o, \theta$ $\theta_a, A, B, V, \alpha, A_1$ $B_1, E, \psi, \phi, \Phi, \chi, T$	quantities defined where used in Sections 1 and 2

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e	fluctuating voltage at the output of a hot wire amplifier
K	constant defined by $e=Ku'$
l	length of the wire
k	constant defined by $K=kl$
$s, s_1, s_2, s_A, s_B, \beta$	variables of integration
$R_x(r)$	*longitudinal correlation coefficient* between the simultaneous turbulent velocity components u' at two points on a line parallel to Ox and a distance r apart
$R_y(r)$	*transverse correlation coefficient* between the simultaneous turbulent velocity components u' at two points on a line parallel to Oy and a distance r apart
L_x, L_y	longitudinal and transverse scales of turbulence
I_b	ratio of the correct to the measured longitudinal intensities of turbulence ($I_b^2=\overline{e^2}/\overline{e_b^2}$ is the ratio of the correct longitudinal turbulent energy to the energy measured with a wire of length l)
λ	microscale of turbulence
G	factor defined by equation (35)
Δ, δ, P, Q	defined with reference to Fig. 3
$R_{b,x}, R_{b,y}$	longitudinal and transverse correlation coefficients as measured with hot wires of length l
$L_{b,x}, L_{b,y}$	longitudinal and transverse scales of turbulence as measured with hot wires of length l
$K_{b,x}, K_{b,y}$	ratios $L_x/L_{b,x}$ and $L_y/L_{b,y}$ respectively
$\lambda_{b,x}, \lambda_{b,y}$	microscales of turbulence found from measurements, with hot wires of length l, of the correlation coefficients $R_{b,x}$ and $R_{b,y}$ respectively

1. Generalities of the Method of Measurement

Let a fine metal wire of diameter d and length l, heated by an electrical current of intensity i, be placed normal to the direction of the velocity V of a gas flow (Fig. 1). If the velocity of the gas is constant, a state of equilibrium is reached between the heat supplied by the electrical current to the wire and the heat lost by convection, radiation, and conduction. The difference between the temperature of the wire θ and the temperature of the ambient air θ_a then remains constant.

The relation between these quantities is given by the equation of L. V. King[1]

$$i^2R=(A+B\sqrt{V})(\theta-\theta_a), \qquad . \quad . \quad . \quad . \quad (1)$$

where A and B are factors depending on the flow conditions, the size and the material of the wire, and the nature of the gas, and where R is the electrical resistance of the wire at the temperature θ. By expressing the temperature

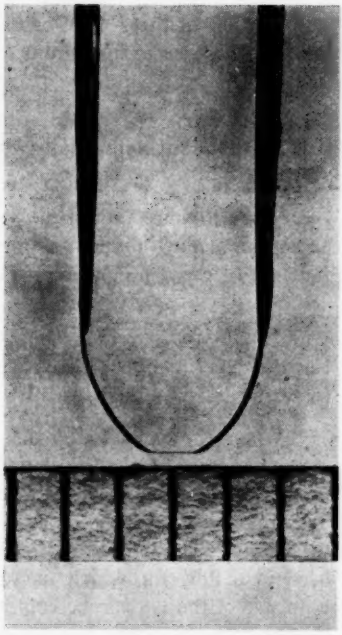

Fig. 1.
A platinum wire fixed at the end of two prongs. The size of the wire can be estimated by comparison with the millimeter scale.

difference as a function of the electrical characteristics of the wire, it is found that

$$\frac{\alpha i^2 R_0 R}{R - R_a} = A + B\sqrt{V} \quad . \quad . \quad . \quad . \quad . \quad (2)$$

with R_0 and R_a the resistances at 0°C. and at the ambient temperature, and α the temperature coefficient of resistance.

Equation (2) gives a relation between the wind velocity, the intensity of the heating current, and the resistance of the wire. One of the last two parameters can be replaced by the potential drop at the terminals of the wire, $E = iR$.

If now the resistance R is kept constant, then the potential drop is given as a function of the velocity by the relation

$$E = \left[\frac{R(R - R_a)}{\alpha R_0}(A + B\sqrt{V})\right]^{\frac{1}{2}} . \quad . \quad . \quad (3)$$

When, on the contrary, the intensity i is kept constant, the voltage is given by

$$E = \frac{(A + B\sqrt{V}) R_a i}{(A + B\sqrt{V}) - \alpha R_0 i^2} . \quad . \quad . \quad . \quad (4)$$

The relationship between the potential drop and the velocity might be modified by including in the electrical circuit a non-linear amplifier. It is possible to obtain in this way the relation

$$E = A_1 + B_1 V, \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where A_1 and B_1 are factors depending on the equipment and the flow conditions.

In each of these three cases the voltage is given as a function of the velocity and may be used to measure it. In general the function

$$E=\psi(V), \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

will be found experimentally by slowly varying the velocity and measuring the voltage.

When the direction of the velocity makes an angle ϕ with the wire, and if this angle is not much different from $\pi/2$, the value of the potential drop is given as a function of the component of the velocity normal to the wire, namely

$$E=\psi(V\sin\phi). \quad . \quad . \quad . \quad . \quad . \quad (7)$$

For the experiments of G. B. Schubauer and P. S. Klebanoff[2] equation (7) is true for values of ϕ as small as 20°.

When the velocity of the flow changes, its fluctuations are not followed instantaneously by the fluctuations of the voltage. There exists a delay due to the thermal lag of the wire. A so-called time constant of the wire describes the magnitude of the time delay. This time constant decreases with the diameter of the wire[3]. The influence of the thermal lag becomes important when the wire is used in a flow for which fluctuations of velocity are of high frequencies. The influence of the lag can be compensated by placing a special electronic stage in the electrical circuit of the wire. The role of the compensation becomes essential when the hot wire is used for measurements of the characteristics of turbulence. The present paper does not deal with the question of compensation, and it is assumed here that the hot-wire equipment has no thermal lag or that its lag is perfectly compensated by an electronic stage.

While we consider here the wire by itself, it should be remembered that it is connected to a heating source. The fluctuations of the velocity changing the electrical conditions of the wire have, therefore, an influence on the heating circuit. For instance, if it is desired to operate at constant current, we allow the resistance of the wire to fluctuate with the velocity of the gas. The overall resistance of the heating circuit of which the resistance of the wire is a part will be affected by these fluctuations[4]. However large the voltage of the source and however small the resistance of the heating current, there may still remain non-negligible fluctuations of the intensity. In the present paper this effect is neglected.

2. Measurement of Turbulent Velocities with a Hot-wire Anemometer

Let $\mathbf{V}(X,Y,Z,t)$ be the instantaneous vectorial velocity of a turbulent fluid at the time t and at a point $P(X,Y,Z)$ defined with respect to stationary orthogonal axes $OXYZ$ (Fig. 2). Let us define the mean velocity at this point by

$$\overline{\mathbf{V}}(X,Y,Z)=\lim_{T\to\infty}\frac{1}{2T}\int_{t-T}^{t+T}\mathbf{V}(X,Y,Z,s)\,ds, \quad . \quad . \quad . \quad (8)$$

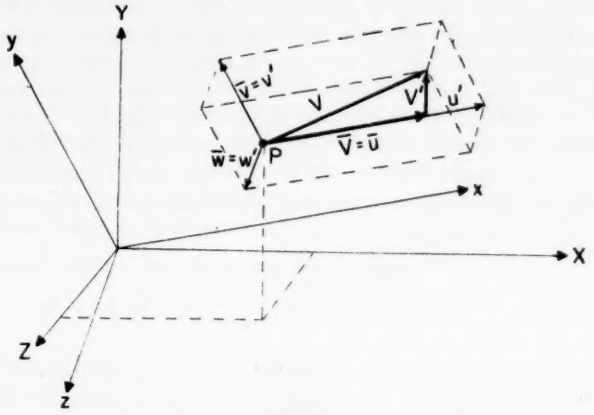

Fig. 2.

where the averaging is made over the time. We assume that, at any time t, there exists at every point of the turbulent field a sub-interval of time $2T$ such that for any interval of time equal to or larger than $2T$ the difference

$$\overline{\mathbf{V}}(X,Y,Z) - \frac{1}{2T}\int_{t-T}^{t+T} \mathbf{V}(X,Y,Z,s)\,ds, \quad . \quad . \quad . \quad . \quad (9)$$

is negligible. We assume, moreover, that the experimental conditions are such that we can consider the gas as being a continuum with a defined instantaneous velocity at every point. These assumptions are acceptable for most of the cases where a hot-wire anemometer is used, for instance, in ordinary wind tunnels.

The instantaneous vectorial velocity at the point $P(X,Y,Z)$ can be divided under these conditions in two parts: $\overline{\mathbf{V}}(X,Y,Z)$ the mean vectorial velocity and $\mathbf{V}'(X,Y,Z,t)$ the turbulent vectorial velocity, and therefore

$$\mathbf{V}(X,Y,Z,t) = \overline{\mathbf{V}}(X,Y,Z) + \mathbf{V}'(X,Y,Z,t). \quad . \quad . \quad . \quad (10)$$

Let there be a second set of orthogonal axes $Oxyz$ with the axis Ox in the direction of the mean velocity $\overline{\mathbf{V}}(X,Y,Z)$. If u, v, w are the components of $\mathbf{V}(x,y,z,t)$ parallel to the axes Ox, Oy, Oz and if for each of these components their mean and their turbulent parts are found, then

$$\left.\begin{aligned} u(x,y,z,t) &= \overline{u}(x,y,z) + u'(x,y,z,t) \\ v(x,y,z,t) &= v'(x,y,z,t) \\ w(x,y,z,t) &= w'(x,y,z,t) \end{aligned}\right\} \quad . \quad . \quad . \quad (11)$$

with $\overline{u} = \overline{V}$ and $\overline{v} = \overline{w} = 0$ due to the choice of the direction of Ox and where $\overline{u'} = \overline{v'} = \overline{w'} = 0$.

There will be around the wire a spatial domain in which the wire integrates, during the measurement, the effect of the velocities at each point of the domain[5]. A rough approximation would be to consider that this domain is equal to the volume of the wire and that the weight factor of each point at this domain is the same. Whatever is the real spatial domain and the distribution of the weight factor, it is assumed in this paper that the turbulence, as well as the mean velocity, is homogeneous in this spatial domain. If, for instance, Φ is an instantaneous value of any fluctuating physical quantity, we would have throughout the domain concerned

$$\overline{\Phi(x, y, z, t)} = \text{constant}. \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The characteristics of turbulence do not change therefore by a translation of the axes.

Consider now a wire placed in a turbulent flow normal to the mean velocity. In the co-ordinate system, the axis Ox is parallel to the direction of the mean velocity and the axis Oy parallel to the axis of the wire. It is assumed in the present section that the dimensions of the wire are negligible and that it measures the velocities at a point. It is assumed also (through the whole present note) that the turbulent velocities are small in comparison with the mean velocity, *i.e.*

$$\bar{u} \gg u', \quad \bar{u} \gg v', \quad \bar{u} \gg w'. \quad . \quad . \quad . \quad . \quad (13)$$

When the angle ϕ is approximately $\pi/2$, equation (7) can be applied. The instantaneous voltage will be, therefore, equal to

$$E = \psi\left(\sqrt{(u^2 + w'^2)}\right),$$

and if we neglect w'^2 in comparison with u^2 it becomes

$$E = \psi(u) = \psi(\bar{u} + u').$$

After expansion in a series and neglecting all the terms except the first two, the last equation gives for the fluctuating voltage

$$E = \psi(\bar{u}) + u' \frac{d\psi(\bar{u})}{d\bar{u}}. \quad . \quad . \quad . \quad . \quad (14)$$

If both sides of (14) are averaged, the mean voltage obtained is

$$\bar{E} = \overline{\psi(\bar{u})}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Under the assumptions of the preceding section, $\psi(\bar{u})$ is independent of the time.

Hence
$$\bar{E} = \psi(\bar{u}) \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

The turbulent voltage is found by subtracting the last equation from equation (14),

and therefore
$$E' = u' \frac{d\psi(\bar{u})}{d\bar{u}}. \quad . \quad . \quad . \quad . \quad . \quad (17)$$

For the measurements of turbulence, the electrical circuit includes an amplifier which, besides compensating the thermal lag of the wire, eliminates the mean

voltage $\bar{E}$ and amplifies the turbulent voltage E'. An ideal amplifier for the hot-wire anemometer, adapted to the explored turbulent flow would:

(*a*) compensate exactly the influence of the thermal lag of the wire,

(*b*) eliminate the mean voltage taken over a time interval at least equal to the time interval over which the average value for the fluctuations of the velocity should be computed, and

(*c*) have a constant factor of amplification and no phase distortion for the entire range of frequencies of the turbulent fluctuations.

At the output of such an amplifier we would measure a fluctuating voltage

$$e=\chi E'=\chi\frac{d\psi(\bar{u})}{d\bar{u}}u', \qquad . \quad . \quad . \quad . \quad (18)$$

where χ would be the factor of amplification.

The derivative $d\psi(\bar{u})/d\bar{u}$ is, in general, a function of the mean velocity $\bar{u}$. As it was assumed that every experiment is made in a field of constant mean velocity, this derivative is considered as remaining constant during the measurement; its value will, however, change when $\bar{u}$ is changed. Equation (18) can be written

$$e=Ku', \qquad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

where $K=\chi d\psi(\bar{u})/d\bar{u}$ is constant for each measurement. The object of the present paper is to discuss the influence of the length of the wire on the measurement of the characteristics of turbulence. In the following sections, after assuming that equation (19) would be verified for a hot-wire of negligible length, a corresponding relation will be written for a wire of non-negligible length. The equations will then be found representing the influence of the length of a hot wire on the measurement of various characteristics of turbulence.

3. Measurement of the Longitudinal Energy of Turbulence with a Hot Wire Normal to the Direction of the Mean Velocity

Let there be a fine wire of negligible diameter and of length l placed in a gaseous flow normal to the direction of the mean velocity. The turbulence of the flow is supposed to be homogeneous and the mean velocity constant in the domain of space-time in which the measurement is made. It is assumed that the thermal lag of the wire is perfectly compensated and that the amplifier eliminates the mean voltage and amplifies by a constant factor the fluctuating voltage for all frequencies included in the turbulence. Consider the orthogonal axes $Oxyz$ with Ox parallel to the direction of the mean velocity and Oy superposed on the wire. The fluid is assumed to be a continuum with defined instantaneous velocities at every point, continuously differentiable with respect to each of the variables x, y, z and t. If under these conditions, an element of length dy of the wire is considered, then, at the time t, it is responsible in the overall output voltage of the amplifier for a part given by

$$\frac{de}{dy}dy=Ku'(y,t)\,dy, \qquad . \quad . \quad . \quad . \quad . \quad (20)$$

with K constant during the experiment. It should be pointed out that equation (20) is, in fact, an approximate relation and its use for a given experimental condition has to be justified. The factor K should be determined from experimental calibration curves.

If the length of the wire is very small in comparison with the scale of turbulence and if the variation of u' along the wire is neglected, then equation (19) is obtained and is rewritten as

$$e(t)=kl\,u'(t)=Ku'(t) \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where $kl=K$ is used to simplify some of the equations which will follow. In the last equation, as in those which follow, the influence of the terminals of the wire, where it is fixed to the supporting prongs, is neglected.

When the length of the wire is not negligible (compared with the scale of turbulence), the variation of $u'(t)$ along the wire cannot be ignored. If $e_b(t)$ represents the instantaneous fluctuating voltage measured with such a wire, then by integrating (20) we find

$$e_b(t)=k\int_0^l u'(s,t)\,ds. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

A thermic ammeter measures fluctuating voltages in terms of root mean square values. The time interval over which the averaging is made depends on the characteristics of the ammeter. It is assumed that this time interval is at least equal to the time interval for which a constant mean velocity may be defined. Various other integrating apparatus beside the thermic ammeter might be used and a regulable averaging time might be provided.

If such averaging apparatus is connected through the amplifier to a wire of negligible length, then one measures

$$\overline{e^2}=k^2l^2\overline{u'^2}=K^2\overline{u'^2}, \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and for a wire of non-negligible length

$$\overline{e_b^2}=k^2\overline{\left[\int_0^l u'(s,t)\,ds\right]^2} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

After transforming the integral given in the last equation into a double integral and remarking that the average of this integral is equal to the integral of the average, it is found that

$$\overline{e_b^2}=k^2\int_0^l\int_0^l \overline{u'(s_1,t)\,u'(s_2,t)}\,ds_1ds_2. \quad . \quad . \quad . \quad . \quad (25)$$

In the last equation there will now be introduced the transverse correlation coefficient which is defined in a homogenous turbulence by the relation

$$R_y(y)=\frac{\overline{u'(x,0,z,t)\,u'(x,y,z,t)}}{\overline{u'^2}}, \quad . \quad . \quad . \quad (26)$$

with $$\overline{u'^2} = \overline{[u'(x,0,z,t)]^2} = \overline{[u'(x,y,z,t)]^2}.$$

Equation (25) in which the coefficient of correlation is introduced, is now written

$$\overline{e_b^2} = k^2 \overline{u'^2} \int_0^l \int_0^l R_y(s_2 - s_1)\, ds_1 ds_2. \quad . \quad . \quad . \quad . \quad (27)$$

If now $$s = s_2 - s_1,$$

and taking into account that R_y is an even function, then

$$\overline{e_b^2} = k^2 \overline{u'^2} \left[\int_0^l ds_1 \int_0^{s_1} R_y(s)\, ds + \int_0^l ds_1 \int_0^{l-s_1} R_y(s)\, ds \right].$$

For each of these double integrals the order of integrations is inverted. After finding the new limits of integrations, one obtains

$$\overline{e_b^2} = 2k^2 \overline{u'^2} \left[\int_0^l ds \int_s^l R_y(s)\, ds_1 + \int_0^l ds \int_0^{l-s} R_y(s)\, ds_1 \right],$$

which leads to $$\overline{e_b^2} = 2k^2 \overline{u'^2} \int_0^l (l-s) R_y(s)\, ds. \quad . \quad . \quad . \quad . \quad (28)$$

The last equation gives the value of $\overline{e_b^2}$ measured with a wire of non-negligible length l, as a function of the true value of $\overline{u'^2}$ and of the transverse correlation curve $R_y(y)$. Comparing this equation to (23), and defining by I_b^2 the factor by which the measured longitudinal turbulent energy* $\overline{e_b^2}$ should be multiplied to give the correct value of this energy, it is found that

$$\frac{1}{I_b^2} = \frac{2}{l^2} \int_0^l (l-s) R_y(s)\, ds, \quad . \quad . \quad . \quad . \quad (29)$$

where $$I_b^2 = \overline{e^2} / \overline{e_b^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (30)$$

As $R_y(y) \leqslant 1$, the turbulent energy measured with a hot wire of non-negligible length is smaller than the true value of this energy. Equation (29) was first found by H. K. Skramstad[9] by a different method and the present author showed it by still another method[7].

3.1. Isotropic turbulence

If the turbulence is homogenous and isotropic, then instead of expressing I_b as a fraction of the transverse correlation coefficient R_y, it is possible to express it as a function of the longitudinal correlation coefficient

$$R_x(x) = \frac{\overline{u'(0,y,z,t)\, u'(x,y,z,t)}}{\overline{u'^2}} \quad . \quad . \quad . \quad . \quad (31)$$

*We call longitudinal energy of turbulence the quantity $\frac{1}{2}\rho u'^2$, where ρ is the density of the gas. By analogy we may speak about $\overline{e_b^2}$ as the measured energy and $\overline{e^2}$ as the real energy, because the values of these measurements lead to a computed value of $\frac{1}{2}\rho\overline{u'^2}$.

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The relation between the two correlations is given by the equation of Th. von Kármán[8]

$$R_y(r)=R_x(r)+\tfrac{1}{2}r\frac{dR_x(r)}{dr}, \quad . \quad . \quad . \quad . \quad (32)$$

where r is the distance between the two points at which the turbulent velocities are considered. The distance r represents here x or y, according to the case. Eliminating R_y between (29) and the last equation, we find the factor of length correction I_b expressed as a function of the longitudinal correlation by

$$\frac{1}{I_b^2}=\frac{1}{l}\int_0^l R_x(s)\,ds. \quad . \quad . \quad . \quad . \quad . \quad (33)$$

Integrating equation (32) from zero to infinity with the condition

$$\lim_{r\to\infty} rR_y(r)=0, \quad . \quad . \quad . \quad . \quad . \quad (34)$$

gives

$$L_x=2L_y, \quad . \quad . \quad . \quad . \quad . \quad . \quad (35)$$

where

$$L_x=\int_0^\infty R_x(s)\,ds, \quad \text{and} \quad L_y=\int_0^\infty R_y(s)\,ds$$

are the longitudinal and transverse scales of turbulence.

3.2. *Approximate relations for large l/L_y*

(*i*) Considering the case when l/L_y is large (and neglecting $yR_y(y)$ and $R_y(y)$ for $y<l$), one finds with equation (28) that

$$\overline{e_b^2}=2k^2\overline{u'^2}\,[lL_y-L_{y(1)}], \quad . \quad . \quad . \quad . \quad . \quad (36)$$

where $L_{y(1)}$ is the first moment of the transverse correlation curve

$$L_{y(1)}=\int_0^\infty sR_y(s)\,ds.$$

It appears, therefore, that when l/L_y is large, the longitudinal energy of the turbulence measured in the flow is a linear function of the length of the wire and decreases with increasing l. The factor I_b can be given for this case by the relation

$$1/I_b^2=2\left[\frac{L_y}{l}-\frac{L_{y(1)}}{L_y^2}\left(\frac{L_y}{l}\right)^2\right], \quad . \quad . \quad . \quad (37)$$

where $L_{y(1)}/L_y^2$ depends on the shape of the correlation curve. When l/L_y increases indefinitely, it is found at the limit

$$\lim_{l/L_y\to\infty}(l/I_b^2)=0. \quad . \quad . \quad . \quad . \quad . \quad (38)$$

and $$\lim_{l/L_y \to \infty} (e_b^2/k^2 l^2) = 0.$$

The longitudinal turbulent energy measured by a wire whose length is very large in comparison with the transverse scale of turbulence will approach zero even if the real energy is not negligible.

(*ii*) In homogenous and isotropic turbulence we can apply equations (33) and (35). For large l/L_y, when $l/L_x = l/(2L_y)$ is also large, we find

$$\overline{e_b}^2 = k^2 \overline{u'^2}\, l L_x = 2k^2 \overline{u'^2}\, l L_y, \quad . \quad . \quad . \quad . \quad . \quad (39)$$

or $$\frac{1}{I_b^2} = \frac{L_x}{l} = \frac{2L_y}{l}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

At large values of l/L_x, the energy of longitudinal turbulence, measured with a wire of non-negligible length in a flow of isotropic turbulence, is inversely proportional to the length of the wire.

3.3. *Approximate relations for small* l/L_y

(*i*) Take now the case when the length of the wire l is small, compared to the transverse scale of turbulence L_y. One will have to consider now the correlation coefficient $R_y(y)$ for small values of y only. Developing in a Taylor's series, we find

$$R_y(y) = 1 + \frac{y^2}{2!}\frac{d^2R_y(0)}{dy^2} + \frac{y^4}{4!}\frac{d^4R_y(0)}{dy^4} + \ldots, \quad . \quad . \quad (41)$$

If all terms of the series except the first three are neglected and if $R_y(y)$ in equation (29) is replaced by its value, then

$$\frac{1}{I_b^2} = 1 + \frac{l^2}{12}\frac{d^2R_y(0)}{dr^2} + \frac{l^4}{360}\frac{d^4R_y(0)}{dr^4}. \quad . \quad . \quad . \quad (42)$$

(*ii*) Consider now the case of isotropic turbulence with the two correlation coefficients $R_x(r)$ and $R_y(r)$ developed in series similar to (41) and introduced in equation (32). Neglecting again the higher order terms we shall find

$$R_y(r) = 1 + r^2\frac{d^2R_x(0)}{dr^2} + \tfrac{1}{8}r^4\frac{d^4R_x(0)}{dr^4}. \quad . \quad . \quad . \quad (43)$$

Using the definition given by G. I. Taylor[9] for the dimension λ of the smallest eddies responsible for dissipation of the turbulent energy

$$\frac{1}{\lambda^2} = -\frac{d^2R_x(0)}{dx^2} = -\tfrac{1}{2}\frac{d^2R_y(0)}{dy^2}, \quad . \quad . \quad . \quad (44)$$

and the notation of G. K. Batchelor and A. A. Townsend[10]

$$G = \lambda^4\frac{d^4R_x(0)}{dx^4}, \quad . \quad . \quad . \quad . \quad (45)$$

one finds for the transverse correlation coefficient the relation

$$R_y(y)=1-\frac{y^2}{\lambda^2}+\frac{1}{8}\frac{G}{\lambda^4}y^4. \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Using the same notations, equation (42) will give

$$\frac{1}{I_b^2}=1-\frac{l^2}{6\lambda^2}+\frac{G}{120}\frac{l^4}{\lambda^4}. \quad . \quad . \quad . \quad . \quad . \quad (47)$$

The last equation could also be found directly from (33).

In (47) the factor I_b is given as a function of l/λ. This equation was found for small values of l/L_y. It is seen now that its use should be considered only for the cases when $l/\lambda<1$. For very small l/λ it might be possible to use even the more simplified relation

$$\frac{1}{I_b^2}=1-\frac{l^2}{6\lambda^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (48)$$

It should be noticed that in both equations (47) and (48) the factor I_b is given as a function of the true dimension λ and not λ_b as it would be measured by a hot wire of non-negligible length. The relation between the true value of λ and the value of λ_b found from measurements with wires of non-negligible length will be considered later. The coefficient G depends, as λ does, on the shape of the correlation curves at small values of r.

3.4. Measurement with a wire whose length l is of the order of magnitude of L_y

When the length of the wire cannot be considered as very large or very small compared with L_y, then it is necessary to use the general equation (29). In that case the factor I_b depends on the shape of the correlation curve.

(*i*) Assuming a transverse correlation function to be of the form

$$R_y(y)=\exp\left(-\frac{|y|}{L_y}\right), \quad . \quad . \quad . \quad . \quad . \quad (49)$$

which was used also by Skramstad[6], it is found that

$$\frac{1}{I_b^2}=2\left\{\frac{L_y}{l}-\frac{L_y^2}{l^2}\left[1-\exp\left(-\frac{l}{L_y}\right)\right]\right\}. \quad . \quad . \quad (50)$$

(*ii*) For a transverse correlation function

$$R_y(y)=\exp\left(-\frac{\pi}{4}\frac{y^2}{L_y^2}\right), \quad . \quad . \quad . \quad . \quad . \quad (51)$$

the value of I_b is given by

$$\frac{1}{I_b^2} = 2\frac{L_y}{l}\operatorname{erf}\left(\frac{\sqrt{\pi}}{2}\frac{l}{L_y}\right) - \frac{4}{\pi}\left(\frac{L_y}{l}\right)^2\left[1-\exp\left(-\frac{\pi}{4}\frac{l^2}{L_y^2}\right)\right], \quad . \quad (52)$$

where $$\operatorname{erf}(a) = \frac{2}{\sqrt{\pi}}\int_0^a \exp(-s^2)\,ds.$$

(*iii*) If now the turbulence is isotropic and the longitudinal correlation function is given by

$$R_x(x) = \left(-\frac{\pi}{4}\frac{x^2}{L_x^2}\right), \quad . \quad . \quad . \quad . \quad (53)$$

then, using equation (33), one finds

$$\frac{1}{I_b^2} = \frac{L_x}{l}\operatorname{erf}\left(\frac{\sqrt{\pi}}{2}\frac{l}{L_x}\right), \quad . \quad . \quad . \quad . \quad (54)$$

or if equation (35) is applied, the factor I_b might be expressed as a function of the transverse scale of turbulence by

$$\frac{1}{I_b^2} = \frac{2L_y}{l}\operatorname{erf}\left(\frac{\sqrt{\pi}}{4}\frac{l}{L_y}\right). \quad . \quad . \quad . \quad . \quad (55)$$

4. Measurement of Correlation Coefficients with Two Hot Wires Normal to the Direction of the Mean Velocity

Two identical wires A and B each having length l and a negligible diameter are placed in a plane making an angle Δ with the direction of the mean velocity U (Fig. 3). The two wires are perpendicular to the direction of the mean velocity and are placed between two planes which are parallel to the direction of U and perpendicular to the plane Δ. Let r be the distance between the two wires, x the projection of r on the direction of the mean velocity, and y its projection on a plane perpendicular to this direction. The two wires are heated to the same mean temperature and the electrical equipment is built in such a manner that to the same fluctuations of the velocity correspond for each wire the same voltage fluctuations. A switch allows the instantaneous potential drops at the terminals of the two wires to be added or subtracted. The resulting voltage is fed into an amplifier which eliminates the mean voltage and amplifies the turbulent voltage. The thermal lag of the wire is assumed to be perfectly compensated. The output of the amplifier is then sent to a meter measuring root-mean-square values of the fluctuating voltage.

Consider first the case when the length l is negligible compared to the transverse scale of turbulence. If the instantaneous fluctuating voltages, e_A, e_B, at the terminals

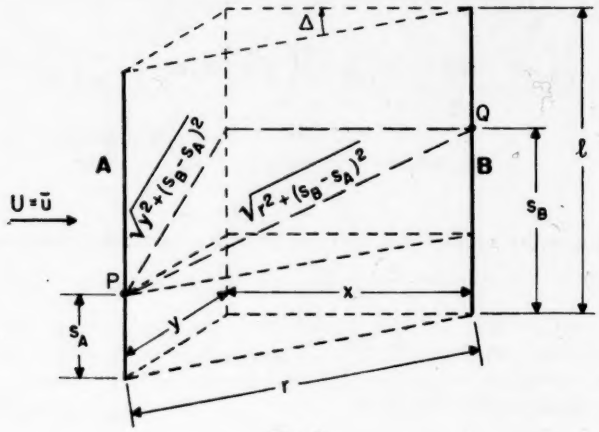

Fig. 3.

of the two wires, A and B, are added, then the averaging meter measures the value of

$$\overline{(e_A+e_B)^2}=k^2l^2\,\overline{(u_A'+u_B')^2} \quad . \quad . \quad . \quad . \quad . \quad (56)$$

where u_A' and u_B' are the longitudinal turbulent velocities at the two wires. If the voltages, e_A, e_B, are opposed, the measure will give

$$\overline{(e_A-e_B)^2}=k^2l^2\,\overline{(u_A'-u_B')^2}. \quad . \quad . \quad . \quad . \quad . \quad (57)$$

Dividing the difference between these two measures by their sum, we obtain

$$\frac{2\overline{e_Ae_B}}{\overline{e_A}^2+\overline{e_B}^2}=\frac{2\overline{u_A'u_B'}}{\overline{u_A'}^2+\overline{u_B'}^2}. \quad . \quad . \quad . \quad . \quad . \quad (58)$$

In a flow of homogeneous turbulence

$$\overline{u'^2}=\overline{u_A'^2}=\overline{u_B'^2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (59)$$

and, in consequence,

$$\overline{e^2}=\overline{e_A}^2=\overline{e_B}^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

Let us represent by $R_\Delta(r)$ the coefficient of correlation along the direction Δ (correlation between the longitudinal turbulent velocities at two points placed at a distance r apart on a straight line which makes an angle Δ with the direction of the mean velocity). This coefficient is found[6], with equations (58) and (60), as

$$R_\Delta(r)=\frac{\overline{u_A'u_B'}}{\overline{u'^2}}=\frac{\overline{e_Ae_B}}{\overline{e^2}}=\frac{\overline{(e_A+e_B)^2}-\overline{(e_A-e_B)^2}}{\overline{(e_A+e_B)^2}+\overline{(e_A-e_B)^2}} \quad . \quad . \quad . \quad (61)$$

Consider now the case where the length of the wires is not negligible in comparison with the scale of turbulence. If $R_{b,\Delta}(r)$ is the correlation coefficient along the direction Δ measured in that case and, $e_{b,A}$, $e_{b,B}$, are the instantaneous potential drops at the terminals of the wires, then

$$R_{b,\Delta}(r)=\frac{\overline{e_{b,A}\,e_{b,B}}}{\overline{e_{b,A}^{\,2}+e_{b,B}^{\,2}}}. \qquad (62)$$

Let P and Q be two points placed on the wires A and B, respectively, at the distances s_A and s_B from the chosen plane of origins (Fig. 3). The instantaneous potential drops, $e_{b,A}$ and $e_{b,B}$, at the terminals of the two wires are given by the relations

$$e_{b,A}=k\int_0^l u'(s_A,t)\,ds_A \quad \text{and} \quad e_{b,B}=k\int_0^l u'(s_B,t)\,ds_B. \qquad (63)$$

Since the turbulence is homogeneous,

$$\overline{[u'(s_A)]^2}=\overline{[u'(s_B)]^2}$$

and

$$\overline{e_{b,A}^{\,2}}=\overline{e_{b,B}^{\,2}}=\overline{e_b^{\,2}}.$$

Equation (62) can then be rewritten

$$R_{b,\Delta}(r)=\frac{\overline{e_{b,A}e_{b,B}}}{\overline{e_b^{\,2}}}. \qquad (64)$$

Applying equation (63), we find

$$\overline{e_{b,A}e_{b,B}}=k^2\overline{\int_0^l u'(s_A,t)\,ds_A\int_0^l u'(s_B,t)\,ds_B}=k^2\int_0^l\int_0^l \overline{u'(s_A,t)\,u'(s_B,t)}\,ds_A ds_B. \qquad (65)$$

Let R_g be the correlation coefficient between the longitudinal turbulent velocities at the two points $P(s_A)$ and $Q(s_B)$. The distance PQ is equal to $\sqrt{\{r^2+(s_B-s_A)^2\}}$ and we have

$$R_g\left(\sqrt{\{r^2+(s_B-s_A)^2\}}\right)=\frac{\overline{u'(s_A,t)\,u'(s_B,t)}}{\overline{u'^2}}, \qquad (66)$$

which will give with equation (65) the relation

$$\overline{e_{b,A}e_{b,B}}=k^2\overline{u'^2}\int_0^l\int_0^l R_g\left(\sqrt{\{r^2+(s_B-s_A)^2\}}\right)ds_A ds_B. \qquad (67)$$

The function R_g is a scalar function of a scalar and a vector. To perform the integration, this function has to be transformed into an ordinary scalar function.

4.1. *Isotropic turbulence*

The transformation of R_δ into an ordinary scalar function can be easily made when the turbulence is homogeneous and isotropic. Applying the equation of Th. von Kármán, we can express R_δ as a function of the longitudinal and of the transverse correlation coefficients R_x and R_y by

$$R_\delta\left(\sqrt{\{r^2+(s_B-s_A)^2\}}\right)=\frac{x^2}{r^2+(s_B-s_A)^2}R_x\left(\sqrt{\{r^2+(s_B-s_A)^2\}}\right)+$$

$$+\frac{y^2+(s_B-s_A)^2}{r^2+(s_B-s_A)^2}R_y\left(\sqrt{\{r^2+(s_B-s_A)^2\}}\right) \qquad (68)$$

If R_δ is replaced in equation (67) by this value, then the following expression is found

$$\overline{e_{b,A}e_{b,B}}=k^2\overline{u'^2}\left[\int_0^l ds_A\int_{-s_A}^{l-s_A}\frac{x^2}{r^2+s^2}R_x(\sqrt{\{r^2+s^2\}})\,ds+\int_0^l ds_A\int_{-s_A}^{l-s_A}\frac{y^2+s^2}{r^2+s^2}R_y(\sqrt{\{r^2+s^2\}})\,ds\right], \qquad (69)$$

where $$s=s_B-s_A.$$

Both coefficients R_x and R_y are even functions of $\sqrt{(r^2+s^2)}$, and the integrals of the last equation can be transformed by operations similar to those which were used in Section 3. After changing the order of integration in the double integrals and finding the new limits of integration, we obtain

$$\overline{e_{b,A}e_{b,B}}=2k^2\overline{u'^2}\left[x^2\int_0^l\frac{l-s}{r^2+s^2}R_x(\sqrt{\{r^2+s^2\}})\,ds+\int_0^l(l-s)\frac{y^2+s^2}{r^2+s^2}R_y(\sqrt{\{r^2+s^2\}})\,ds\right].$$

The correlation coefficient $R_{b,\Delta}$ measured with the wires of non-negligible length can now be obtained as a function of the real longitudinal and transverse correlation coefficients R_x and R_y from equations (23), (64), and the last equation. We find

$$R_{b,\Delta}(r)=\frac{2I_b^2}{l^2}\left[x^2\int_0^l\frac{l-s}{r^2+s^2}R_x(\sqrt{\{r^2+s^2\}})\,ds+\int_0^l(l-s)\frac{y^2+s^2}{r^2+s^2}R_y(\sqrt{\{r^2+s^2\}})\,ds\right], \qquad (70)$$

where the factor I_b is given by equation (30).

(*i*) It is, of course, possible to eliminate one of the functions R_x or R_y from equation (70) by applying the von Kármán equation (32).

(*ii*) When the two wires are placed in a plane parallel to the direction of the mean velocity ($\Delta=0°$) with their axes perpendicular to this direction, then $y=0$ in

equation (70). Eliminating now R_y between equations (32) and (34), we express the measured longitudinal correlation as a function of the real longitudinal correlation function by

$$R_{b,x}(x)=\frac{I_b^2}{l}\int_0^l R_x(\sqrt{\{x^2+s^2\}})\,ds, \quad . \quad . \quad . \quad (71)$$

where I_b is given by equation (33).

4.2. *Axi-symmetric turbulence*

The case of axi-symmetric turbulence with the axis of symmetry parallel to the direction of the mean velocity includes the special case of isotropic turbulence. The relation found hereafter is, therefore, also valid for isotropic turbulence. For the measurement of the correlation coefficient, the two wires are placed in a plane perpendicular to the direction of the mean velocity ($\Delta=\pi/2$). In this case $x=0$ in equation (70), and we find for the measured transverse correlation coefficient

$$R_{b,y}(y)=\frac{2I_b^2}{l^2}\int_0^l (l-s)\,R_y(\sqrt{\{y^2+s^2\}})\,ds, \quad . \quad . \quad . \quad (72)$$

with the factor I_b given by equation (29). This equation was found first by H. K. Skramstad[6] by quite a different method.

In equations (70) to (72) the correlation coefficients, measured with hot wires of non-negligible length l are expressed as functions of this length, of the correct correlation function, and of the length correction factor I_b. The factor I_b is itself given as a function of l and is a function of $R(r)$ through equations (29) or (33), according to the case. In order to find $R_b(r)$, it is necessary, therefore, to know the form of the function $R(r)$. The shape of the correlation curve varies with the experimental conditions, and various equations will have to be used to represent this curve. It is often difficult to represent the correlation function by an equation which gives values of $R(r)$ in good agreement with the correlation coefficients over a large interval 0, r and is simple enough for application in equations (70) to (72). It is in general easier to represent the experimental correlation curve by two equations: one for small values of r and another for r larger than some given value r_1. The factor I_b is given as a function of $R(r)$ in the interval 0, l, while the integrals of equations (70) to (72) are functions of $R(r)$ in the interval r, $\sqrt{(r^2+l^2)}$. We should try, therefore, to use an equation which accurately represents $R(r)$ in the interval 0, l to compute I_b and an equation accurately representing the correlation curve in the interval r, $\sqrt{(r^2+l^2)}$ for equations (70) to (72). It will evidently be preferable to represent the correlation curve by a single equation if it is possible to do so without excessive difficulties.

4.3. *Approximate relations for large r/l*

When the length l of the wire is small compared with the distance r between the two points at which the longitudinal turbulent velocities are considered and this distance is not small compared with the scale of turbulence, then equations (71) and (72) can be simplified. As the interval of integration is $0 \leqslant s \leqslant l$, we find by neglecting the terms with s^6/r^6 and higher that

$$\sqrt{(r^2+s^2)} = \left| \left(r + \frac{1}{2}\frac{s^2}{r} - \frac{1}{8}\frac{s^4}{r^3} \right) \right| .$$

Developing $R(\sqrt{\{r^2+s^2\}})$ in a Taylor's series and neglecting the terms with s^6/r^6 and higher, we find

$$R(\sqrt{\{r^2+s^2\}}) = R(r) + \frac{1}{2}\frac{s^2}{r}R'(r) + \frac{1}{8}\frac{s^4}{r^2}\left[R''(r) - \frac{1}{r}R'(r)\right] . \quad . \quad (73)$$

For the case of isotropic turbulence, equations (71) and (73) give the measured longitudinal correlation coefficient

$$R_{b,x}(x) = I_b^2 \left\{ R_x(x) + \frac{1}{6}\frac{l^2}{x}R_x'(x) + \frac{1}{40}\frac{l^4}{x^2}\left[R_x''(x) - \frac{1}{x}R_x'(x)\right] \right\} , \quad (74)$$

where the factor I_b is given by equation (48) or (49) according to the magnitude of l/λ. The last equation can also be written as follows

$$R_x(x) = \frac{1}{I_b^2}R_{b,x}(x) - \frac{l^2}{6x}R_x'(x) + \frac{l^4}{40x^2}\left[\frac{1}{x}R_x'(x) - R_x''(x)\right] . \quad . \quad (75)$$

In this equation the correct coefficient of correlation R_x is given as a function of $R_{b,x}$ measured with hot wires of non-negligible length and of the values of R_x'. As a first approximation, we can use in equation (75) the measured values $R_{b,x}'(x)$ and $R_{b,x}''(x)$ instead of the correct values $R'(x)$ and $R''(x)$.

For very small values of l/x, the terms with l^4 can be neglected. At very large distances x, when l/x is still smaller, we shall have simply

$$R_x(x) = \frac{1}{I_b^2}R_{b,x}(x) . \quad . \quad . \quad . \quad . \quad . \quad (76)$$

In a simpler way for the case of axi-symmetric turbulence equation (72) gives with (73) for the measured transverse correlation coefficient

$$R_{b,y}(y) = I_b^2 \left\{ R_y(y) + \frac{1}{12}\frac{l^2}{y}R_y'(y) + \frac{1}{120}\frac{l^4}{y^2}\left[R_y''(y) - \frac{1}{y}R_y'(y)\right] \right\} ,$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad (77)$$

and $$R_y(y)=\frac{1}{I_b^2}R_{b,y}(y)-\frac{1}{12}\frac{l^2}{y}R_y'(y)+\frac{1}{120}\frac{l^4}{y^2}\left[\frac{1}{y}R'(y)-R_y''(y)\right]. \qquad (78)$$

The last equation might be used to correct the measured values of the transverse correlation coefficient if, as a first approximation, $R_{b,y}'(y)$ and $R_{b,y}''(y)$ are used in (78) instead of $R_y'(y)$ and $R_y''(y)$.

When l/y is sufficiently small we can use the simplified equation

$$R_y(y)=\frac{1}{I_b^2}R_{b,y}(y). \qquad (79)$$

4.4. *Influence on the shape of the measured correlation curve*

The value of R_b, as measured with hot wires of non-negligible length, can be computed if we assume some given function to represent the correct correlation curve.

In axi-symmetric turbulence with a transverse correlation function of the form

$$R_y(y)=\exp(-|y|/L_y). \qquad (80)$$

the corresponding $R_{b,y}(y)$ values were computed numerically by H. K. Skramstad[6]. When the correlation function is of the form

$$R_y(y)=\exp\left(-\frac{\pi}{4}\frac{y^2}{L_y^2}\right), \qquad (81)$$

we find by applying (72)

$$R_{b,y}(y)=2I_b^2\frac{L_y}{l}\left\{\operatorname{erf}\left(\frac{\sqrt{\pi}}{2}\frac{1}{L_y}\right)-\frac{2}{\pi}\frac{L_y}{l}\left[1-\exp\left(\frac{\pi}{4}\frac{l^2}{L_y^2}\right)\right]\right\}\exp\left(-\frac{\pi}{4}\frac{y^2}{L_y^2}\right).$$

Replacing I_b by its value given by equation (52), gives

$$R_{b,y}(y)=R_y(y). \qquad (82)$$

Consider now the case of isotropic turbulence with the longitudinal correlation function

$$R_x(x)=\exp\left(-\frac{\pi}{4}\frac{x^2}{L_x^2}\right). \qquad (83)$$

Eliminating R_x and I_b between the last equation and equations (35) and (54) gives

$$R_{b,x}(x)=R_x(x). \qquad (84)$$

The transverse correlation function corresponding to (83) is found with (32). We obtain

$$R_y(y)=\left(1-\frac{\pi}{16}\frac{y^2}{L_y^2}\right)\exp\left(-\frac{\pi}{16}\frac{y^2}{L_y^2}\right), \quad . \quad . \quad (85)$$

and applying equations (72) and (55), we find for the transverse correlation measured with wires of non-negligible length

$$R_{b,y}(y)=\left(1-A\frac{\pi}{16}\frac{y^2}{L_y^2}\right)\exp\left(-\frac{\pi}{16}\frac{y^2}{L_y^2}\right), \quad . \quad . \quad (86)$$

where

$$A=2-\frac{8}{\pi}\frac{1-\exp\left(-\frac{\pi}{16}\frac{l^2}{L_y^2}\right)}{\frac{l}{L_y}\operatorname{erf}\left(\frac{\sqrt{\pi}}{4}\frac{l}{L_y}\right)}.$$

5. Measurement of the Scale of Turbulence

The scale of turbulence (longitudinal or transverse) is found by taking the integral of the corresponding correlation curve from zero to infinity

$$L=\int_0^\infty R(s)\,ds. \quad . \quad . \quad . \quad . \quad . \quad (87)$$

In general, the value of L cannot be obtained with great precision because the upper limit of integration is infinity.

When the length of the wires is not negligible, the measured scale of turbulence is equal to

$$L_b=\int_0^\infty R_b(s)\,ds. \quad . \quad . \quad . \quad . \quad . \quad (88)$$

The factors by which the measured values of the longitudinal and transverse scales of turbulence $L_{b,x}$ and $L_{b,y}$ should be multiplied to give the real scales of turbulence L_x and L_y are

$$K_{b,x}=\frac{L_x}{L_{b,x}} \quad \text{and} \quad K_{b,y}=\frac{L_y}{L_{b,y}}. \quad . \quad . \quad . \quad (89)$$

(*i*) For axisymmetric turbulence with a transverse correlation function of the form

$$R_y(y)=\exp(-|y|/L_y),$$

it is found[6] that

$$K_{b,y}=2I_b^2\frac{L_y^2}{l^2}\left[\frac{\pi}{2}\frac{l}{L_y}-2+\frac{L}{L_y}J\left(\frac{l}{L_y}\right)\right]. \quad . \quad . \quad (90)$$

where

$$J(a)=\int_0^{\pi/2}\left(1+\frac{2}{a\operatorname{cosec}\beta}\right)\exp(-a\operatorname{cosec}\beta)\,d\beta, \quad . \quad . \quad (91)$$

and where I_b is given by equation (50).

(*ii*) When the transverse correlation function is

$$R_y(y)=\exp\left(-\frac{\pi}{4}\frac{y^2}{L_y^2}\right),$$

we find that $K_{b,y}=1$ and $L_{b,y}=L_y$. . . . . (92)

(*iii*) If we consider the case of isotropic turbulence with a longitudinal correlation function

$$R_x(x)=\exp\left(-\frac{\pi}{4}\frac{x^2}{L_x^2}\right),$$

then $K_{b,x}=1$ and $L_{b,x}=L_x$. . . . . (93)

Applying equation (50) we find for this case that

$$K_{b,y}=\frac{8}{\pi}\frac{1-\exp\left(-\frac{\pi}{16}\frac{l^2}{L_y^2}\right)}{\frac{l}{L_y}\operatorname{erf}\left(\frac{\sqrt{\pi}}{4}\frac{l}{L_y}\right)}. \quad . \quad . \quad . \quad (94)$$

6. Measurement of the Microscale of Turbulence

The Taylor dimension λ of the smallest eddies responsible for the dissipation of the turbulent energy, which is often called the microscale of turbulence, is a very important characteristic of isotropic turbulence. The dimension λ is much better defined theoretically than the scale of turbulence, being given simply by the relation

$$\lambda^2=-\frac{1}{R_x''(0)}=-\frac{2}{R_y''(0)}, \quad . \quad . \quad . \quad . \quad (95)$$

From equations (42) and (43) we find for small distances r the relations

$$R_x(x)=1-\frac{x^2}{2\lambda^2}+\frac{G}{24}\frac{x^4}{\lambda^4} \quad . \quad . \quad . \quad . \quad . \quad (96)$$

$$R_y(y)=1-\frac{y^2}{\lambda^2}+\frac{G}{8}\frac{y^4}{\lambda^4} \quad . \quad . \quad . \quad . \quad . \quad (97)$$

of which the second was given before as equation (46). We shall consider the two cases: (*a*) when the microscale of turbulence is obtained from the longitudinal correlation curve and (*b*) when it is found by using the transverse correlation curve. It should be remembered that in the present paper we consider only the influence of the non-negligible length of the wires on the measurements, assuming that from every other viewpoint the hot-wire equipment is perfect. For instance, we neglect for the present the effect of the wake of one wire on another, which may be quite important when R_x for small x is measured. We neglect here also the effect of

radiation between the wires, and of the perturbation of the flow, which may not be negligible when the distance between the wires is very small.

Consider now the case when the microscale of turbulence is measured with two hot wires of non-negligible length placed in a plane parallel to the direction of the mean velocity. The measured $\lambda_{b,x}$ will be defined by the relation

$$\lambda_{b,x}^2 = -\frac{1}{R_{b,x}''(0)}. \qquad (98)$$

From equation (71),

$$\frac{1}{\lambda_{b,x}^2} = -\frac{I_b^2}{l}\int_0^l \frac{R_x'(s)}{s}\,ds. \qquad (99)$$

If the microscale of turbulence is measured with two wires of non-negligible length placed in a plane perpendicular to the direction of the mean velocity, then the measured $\lambda_{b,y}$ is defined by the relation

$$\lambda_{b,y}^2 = -\frac{2}{R_{b,y}''(0)}, \qquad (100)$$

and, from equation (72),

$$\frac{1}{\lambda_{b,y}^2} = -\frac{I_b^2}{l^2}\int_0^l \frac{(l-s)}{s} R_y'(s)\,ds. \qquad (101)$$

When the turbulence is assumed to be isotropic, then equation (32) can be applied and we obtain, with equation (101),

$$\frac{1}{\lambda_{b,y}^2} = -\frac{I_b^2}{l}\left[\frac{3}{2}\int_0^l \frac{R_x'(s)}{s}\,ds + \frac{1-R_x(l)}{l}\right]. \qquad (102)$$

In this relation the microscale of turbulence measured with two parallel wires placed in a transverse plane is given as a function of the longitudinal correlation coefficient $R_x(x)$.

We find, moreover, for the case of isotropic turbulence the relation

$$\frac{\lambda_{b,x}^2}{\lambda_{b,y}^2} = 1 + \left[\frac{1}{2} - \frac{\int_0^l \frac{R_x'(s)}{l}\,ds}{\int_0^l \frac{R_x(s)}{s}\,ds}\right] \qquad (103)$$

where the expression in the brackets tends to zero with l. The last equation shows that the value of the microscale of isotropic turbulence, measured with two wires of non-negligible length placed in a plane parallel to the direction of the mean velocity, is different from the microscale measured with wires placed in a plane perpendicular to the direction of the mean velocity.

When the length of the wire is small compared with λ, then equations (96) and (97) may be used to represent the correlation coefficient. In what follows we shall neglect the terms with $(l/\lambda)^4$.

When λ is obtained from measurements with two wires of non-negligible length placed in a plane parallel to the direction of the mean velocity, then we can apply equation (99). After finding R_x' from equation (96) and replacing I_b by its value given in (48), we have

$$\frac{\lambda^2}{\lambda_{b,x}^2} = \frac{18 - G\left(\frac{l}{\lambda}\right)^2}{18 - 3\left(\frac{l}{\lambda}\right)^2}. \qquad (104)$$

This relation gives the factor by which the measured value of $\lambda_{b,x}^2$ should be multiplied to give the correct value λ^2. We shall notice that $\lambda_{b,x} = \lambda$ for $G = 3$ and that the real λ is smaller than the measured $\lambda_{b,x}$ for $G > 3$, which in practice seems to be the most general case. Instead of being expressed as a function of (l/λ) the correction factor $(\lambda/\lambda_{b,x})$ can be given as a function of $(l/\lambda_{b,x})$. Solving equation (104) for λ gives

$$\frac{\lambda^2}{\lambda_{b,x}^2} = \frac{1}{2}\left\{\left(\frac{1}{6}\frac{l^2}{\lambda_{b,x}^2} + 1\right) + \sqrt{\left[\left(\frac{1}{6}\frac{l^2}{\lambda_{b,x}^2} + 1\right)^2 - \frac{2}{9}G\frac{l^2}{\lambda_{b,y}^2}\right]}\right\}. \qquad (105)$$

If now the microscale of turbulence is obtained from measurements with two parallel wires of non-negligible length placed in a plane perpendicular to the direction of the mean velocity, then we apply one of the two equations (101) or (102). It will be found that

$$\frac{\lambda^2}{\lambda_{b,y}^2} = \frac{24 - G\left(\frac{l}{\lambda}\right)^2}{24 - 4\left(\frac{l}{\lambda}\right)^2} \qquad (106)$$

The real λ is smaller than the measured $\lambda_{b,y}$ as long as $G > 4$. If we express the correction factor $(\lambda/\lambda_{b,y})$ as a function of $(l/\lambda_{b,y})$, we find

$$\frac{\lambda^2}{\lambda_{b,y}^2} = \frac{1}{2}\left\{\left(\frac{1}{6}\frac{l^2}{\lambda_{b,y}^2} + 1\right) + \sqrt{\left[\left(\frac{1}{6}\frac{l^2}{\lambda_{b,y}^2} + 1\right)^2 - \frac{G}{6}\frac{l^2}{\lambda_{b,y}^2}\right]}\right\}. \qquad (107)$$

The ratio of the microscale of turbulence measured in a plane parallel to the mean velocity direction to the one measured in plane perpendicular to this direction is given by

$$\frac{\lambda_{b,x}^2}{\lambda_{b,y}^2} = \frac{1 - \frac{G}{24}\left(\frac{l}{\lambda}\right)^2}{1 - \frac{G}{18}\left(\frac{l}{\lambda}\right)^2}. \qquad (108)$$

When the distance r is small, we find (for isotropic turbulence), using equations (96) and (97), the following simple relation between the longitudinal correlation coefficient $R_x(x)$ and the transverse correlation coefficient $R_y(y)$

$$\frac{1-R_y(r)}{1-R_x(r)}=2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (109)$$

The similar relation for the correlation coefficients measured with wires of non-negligible length will be

$$\frac{1-R_{b,y}(r)}{1-R_{b,x}(r)}=2\frac{\lambda_{b,x}{}^2}{\lambda_{b,y}{}^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (110)$$

This ratio is larger than 2.

ACKNOWLEDGMENT

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On a Simple Method for Calculating Laminar Boundary Layers

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Summary: A simple one parametric method, due to A. Walz and based on the momentum and energy equations, for calculating approximately laminar boundary layers is extended to cover axi-symmetric flow as well as plane flow. The necessary computing work is reduced a little.

Another known method which requires still less computing work is also extended for axi-symmetric flow and, with the amendment of a numerical constant, proves adequate for practical purposes.

1. Introduction

Since there are already several methods of calculating approximately laminar boundary layers for incompressible flow, it would seem necessary to justify the development of yet another method. The following method tries to obtain in a practical manner, results as correct as possible and with the smallest amount of computing work. Obviously a very accurate method is to represent the velocity profiles approximately by a two parametric class whereby the partial differential equation for boundary layers is replaced by two ordinary differential equations: the momentum and the energy equation[1]. But here the interpolation in the two parameters requires considerably more computing work than that with the usual one parametric methods. Therefore A. Walz[2] altered this method back again into a one parametric method using the energy equation instead of the boundary condition for the second derivative $\partial^2 u/\partial y^2$ on the wall. By this means he obtained better results than by the usual one parametric methods based on the momentum equation only. So he proved that it is more important to satisfy the energy balance on the average over the whole layer rather than the usual boundary condition for the curvature of the velocity profile at the wall. As the latter condition is too sharp for a limited variety of profiles, a wrong profile is sometimes selected by it. The method of A. Walz can be altered slightly as follows. It is extended so as to calculate boundary layers on axi-symmetric bodies (without incidence) as well as in two-dimensional plane flow. Also, the computing work is reduced.

Notation

x the axial co-ordinate

r the radial co-ordinate

r_0 the co-ordinate of the body of revolution

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s the distance measured along the surface of the body

n the distance normal to the surface

$\frac{r}{r_0} = 1 + \frac{r_0}{n \cos \alpha}$ with $\tan \alpha = \frac{d r_0}{dx}$

u, v the normal velocity components at any point in the layer

U the velocity at the edge of the boundary layer

U_0 the velocity of the undisturbed stream

ρ the density of the incompressible fluid

μ the viscosity

$\nu = \mu/\rho =$ the kinematic viscosity

δ the thickness of the boundary layer

$\delta_1 = \int_0^\infty \left(1 - \frac{u}{U}\right) \frac{r}{r_0} dn =$ the displacement thickness

$\delta_2 = \int_0^\infty \frac{u}{U} \left(1 - \frac{u}{U}\right) \frac{r}{r_0} dn =$ the momentum thickness

$\delta_3 = \int_0^\infty \frac{u}{U} \left[1 - \left(\frac{u}{U}\right)^2\right] \frac{r}{r_0} dn =$ the energy thickness

$D = \int_0^\infty \left(\frac{\partial (u/U)}{\partial (n/\delta_2)}\right)^2 \frac{r}{r_0} d\left(\frac{n}{\delta_2}\right) =$ the dimensionless dissipation in the layer

$H = \delta_1/\delta_2 =$ a parameter characterising the velocity profile

$H_{32} = \delta_3/\delta_2$

$R_2 = U \delta_2/\nu =$ the local Reynolds number of the boundary layer

$R = U_0 R'/\nu =$ the Reynolds number of the main stream

R' a characteristic length

$X = \delta_2 R_2 = U \delta_2^2/\nu =$ a useful computing quantity

$\tau_0 = \mu \left(\frac{\partial u}{\partial n}\right)_{n=0} =$ the shear stress at the wall = the skin friction

$\varepsilon = \left(\frac{\partial (u/U)}{\partial (n/\delta_2)}\right)_0 = \frac{\tau_0}{\rho U^2} R_2 =$ the dimensionless skin friction

$\lambda = -\frac{\delta_2^2}{U} \left(\frac{\partial^2 u}{\partial n^2}\right)_0 =$ another characteristic profile parameter

2. The Calculation Method

As the equations for axi-symmetric flow include those for plane flow and are only a little more complicated, the method may be derived for the boundary layer along a body of revolution, the co-ordinates of which are x (along the axis) and the radius $r_0(x)$. In the case of plane flow all the equations are to be simplified by putting $r_0(x) = \text{constant} \longrightarrow \infty$.

The equation of continuity is

$$\frac{\partial}{\partial s}(ru) + \frac{\partial}{\partial n}(rv) = 0 \qquad (1)$$

and the boundary layer equation

$$u\frac{\partial u}{\partial s} + v\frac{\partial u}{\partial n} = U\frac{dU}{ds} + \nu\frac{\partial^2 u}{\partial n^2}. \qquad (2)$$

From the latter, two ordinary differential equations can be derived by multiplying by r or ur and integrating through the layer. These are the momentum equation

$$\nu r_0\left(\frac{\partial u}{\partial n}\right)_0 = \frac{d}{ds}\int_0^\infty u(U-u)\,rdn + \frac{dU}{ds}\int_0^\infty (U-u)\,rdn \qquad (3)$$

and the energy equation

$$\nu\int_0^\infty \left(\frac{\partial u}{\partial n}\right)^2 rdn = \frac{d}{ds}\int_0^\infty u\left(\tfrac{1}{2}U^2 - \tfrac{1}{2}u^2\right)rdn. \qquad (4)$$

The physical meaning of the latter equation is that the energy dissipated by the viscosity equals the loss of kinetic energy inside the whole layer.

Equations (3) and (4) can be written more shortly by introducing the quantities defined. Then equation (3) becomes

$$\frac{dX}{ds} + \left\{(3+2H)\frac{1}{U}\frac{dU}{ds} + \frac{2}{r_0}\frac{dr_0}{ds}\right\}X - 2\varepsilon = 0 \qquad (5)$$

This is the usual differential equation for X or δ_2, the momentum thickness. Another equation for the profile parameter H can be derived by combining equations (3) and (4) as follows:

$$-\frac{dH}{ds} = f\frac{1}{U}\frac{dU}{ds} + g\frac{1}{X}, \qquad (6)$$

where

$$f = -\frac{H_{32}(H-1)}{dH_{32}/dH} \quad \text{and} \quad g = \frac{\varepsilon H_{32} - 2D}{dH_{32}/dH}. \qquad (7)$$

TABLE I

H	A	B	C	D	E	F	G	ε
2	−1·11	−0·83	6·4	0·73	2·2	7·9	0·26	0·465
	4	0·25	0·8	6	0·3	0·8	9	26
2·05	−1·07	−0·58	7·2	0·79	2·5	8·7	0·35	0·439
	4	0·27	0·9	7	0·4	0·8	0·10	25
2·1	−1·03	−0·31	8·1	0·86	2·9	9·5	0·45	0·414
	5	0·30	1·0	7	0·4	0·9	0·10	24
2·15	−0·98	−0·01	9·1	0·93	3·3	10·4	0·55	0·390
	6	0·34	1·1	9	0·5	1·0	0·10	23
2·2	−0·92	+0·33	10·2	1·02	3·8	11·4	0·65	0·367
	7	0·39	1·2	9	0·5	1·1	0·11	22
2·25	−0·85	0·72	11·4	1·11	4·3	12·5	0·76	0·345
	8	0·45	1·3	0·10	0·5	1·3	0·13	21
2·3	−0·77	1·17	12·7	1·21	4·8	13·8	0·89	0·324
	9	0·51	1·4	0·11	0·6	1·5	0·13	20
2·35	−0·68	1·68	14·1	1·32	5·4	15·3	1·02	0·304
	0·11	0·59	1·6	0·12	0·7	1·6	0·14	19
2·4	−0·57	2·27	15·7	1·44	6·1	16·9	1·16	0·285
	0·13	0·66	1·8	0·14	0·7	1·8	0·15	18
2·45	−0·44	2·93	17·5	1·58	6·8	18·7	1·31	0·267
	0·15	0·74	2·0	0·15	0·8	1·9	0·17	17
2·5	−0·29	3·67	19·5	1·73	7·6	20·6	1·48	0·250
	0·17	0·82	2·2	0·16	0·9	2·1	0·18	16
2·55	−0·12	4·49	21·7	1·89	8·5	22·7	1·66	0·234
	0·18	0·90	2·4	0·18	1·0	2·4	0·19	15
2·6	+0·06	5·39	24·1	2·07	9·5	25·1	1·85	0·219
	0·19	0·99	2·7	0·20	1·0	2·6	0·22	15
2·65	0·25	6·38	26·8	2·27	10·5	27·7	2·07	0·204
	0·21	1·09	2·9	0·22	1·1	2·9	0·23	14
2·7	0·46	7·47	29·7	2·49	11·6	30·6	2·30	0·190
	0·23	1·21	3·2	0·24	1·2	3·2	0·25	13
2·75	0·69	8·68	32·9	2·73	12·8	33·8	2·55	0·177
	0·26	$1{\cdot}3_2$	3·5	0·26	1·4	3·6	0·28	12
2·8	0·95	10·0	36·4	2·99	14·2	37·4	2·83	0·165
	0·29	1·5	3·9	0·28	1·5	4·0	0·29	11
2·85	1·24	11·5	40·3	3·27	15·7	41·4	3·12	0·154
	0·32	1·7	4·3	0·30	1·7	4·4	0·31	11
2·9	1·56	13·2	44·6	3·57	17·4	45·8	3·43	0·143
	0·36	1·9	4·7	0·32	2·0	4·8	0·33	10
2·95	1·92	15·1	49·3	3·89	19·4	50·6	3·76	0·133
	0·40	2·0	5·2	0·34	2·2	5·3	0·35	10
3·0	2·32	17·1	54·5	4·23	21·6	55·9	4·11	0·123

These equations are still exact, of course, in general it would be impossible to compute the boundary layer only by equations (5) and (6), as these two ordinary differential equations cannot replace the partial differential equation (2) and all its boundary conditions. However, restricting oneself to an approximate calculation by assuming a one parametric class of velocity profiles, H_{32}, ε and D and consequently f and g become functions of the one parameter H alone: $f(H)$ and $g(H)$. In this case obviously equations (5) and (6) enable $X(s)$ and $H(s)$ to be computed if the body $r_0(s)$ and its pressure distribution, or $U(s)$, are given.

For axi-symmetric flow this assumption of a one parametric class implies neglecting the term r/r_0 during the integration across the layer (*see* the definitions of the characteristic quantities). This is the same approximation Mangler [3] made in his paper about the transformation of an axi-symmetric boundary layer into a corresponding plane one. Since usually the layer is very thin when compared with the radius of the body of revolution, the assumption $r/r_0=1$ is good enough as r varies only between r_0 and $r_0+\delta$. The influence of the shape of the body of revolution still remains in the term $(2/r_0)(dr_0/ds)X$ in equation (5), whereas equation (6) is approximately the same whether the flow is axi-symmetric or plane, as it is independent of r_0.

The system of the two differential equations (5) and (6) is to be solved step by step. For that the step Δs may be chosen so small that $(1/U)\,dU/ds$ can be taken as constant through the integration step. Then the local pressure gradient is characterised by

$$\gamma=\frac{U_2-U_1}{(U_1+U_2)/2}\approx\frac{1}{U}\frac{dU}{ds}\Delta s=-\frac{1}{\rho U^2}\frac{dp}{ds}\Delta s, \qquad . \quad . \quad (8)$$

where the index 1 denotes the beginning and 2 the end of the interval. Practically this means that the steps Δs must be made as small as is necessary to draw the curve $U(s)$ with reasonable accuracy.

In the same way all the other functions are linearised over the interval. Hence from equation (6)

$$X_2=-\frac{2(g+\frac{1}{2}\dot{g}\Delta H)}{\Delta H+(f+\frac{1}{2}\dot{f}\Delta H)\gamma}\Delta s-X_1, \quad \text{where} \quad \Delta H=H_2-H_1$$

$$\dot{f}=\frac{df}{dH} \quad \text{and} \quad \dot{g}=\frac{dg}{dH}. \qquad . \quad . \quad . \quad (9)$$

Introducing this expression for X in equation (5) a quadratic equation for ΔH is obtained. Yet for small steps Δs, ΔH will also be small so that the term with $(\Delta H)^2$ can be neglected. Then

$$\Delta H=-\frac{A(1+\omega)+B\gamma+C\dfrac{X_1}{\Delta s}\gamma}{D+E\gamma+F\dfrac{X_1}{\Delta s}\gamma+\dfrac{X_1}{\Delta s}+G\omega}. \qquad . \quad . \quad (10)$$

where

$$\omega=\frac{1}{r_0}\frac{dr_0}{ds}\Delta s\approx\frac{(r_0)_2-(r_0)_1}{[(r_0)_1+(r_0)_2]/2}$$

($\omega=0$ for plane flow), and A, B, ... G are the following functions of H:—

$$A=g, B=\left(\frac{3}{2}+H\right)g+\varepsilon f, C=f, D=\frac{1}{2}\dot{g}+\varepsilon, E=\frac{1}{2}\dot{B}, F=\frac{1}{2}\dot{f}, G=\frac{1}{2}\dot{g}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

In equation (10) the argument H in these functions is to be taken at the beginning of the step: $A(H_1)$, $B(H_1)$, ... $G(H_1)$. Once ΔH is computed ΔX follows from an equation derived from equation (5):

$$\Delta X=\frac{\varepsilon_1+\varepsilon_2-[(H_1+H_2+3)\gamma+2\omega]X_1/\Delta s}{1+\frac{1}{2}[(H_1+H_2+3)\gamma+2\omega]}\Delta s. \quad . \quad (12)$$

According to its definition X gives the momentum thickness

$$\delta_2{}^2=X\nu/U \quad \text{or} \quad \frac{\delta_2}{R'}\sqrt{R}=\sqrt{\left(\frac{X/R'}{U/U_0}\right)}, \quad . \quad . \quad (13)$$

where $R=U_0R'/\nu$ is the Reynolds number of the undisturbed stream and R' a characteristic length. With δ_2 giving the size and H the shape of the velocity profile the boundary layer is known once $X(s)$ and $H(s)$ are calculated.

To establish the functions $A, B, \ldots G$ a certain class of velocity profiles has to be chosen. Walz[2] has shown that the results do not depend very much on the particular class if the energy equation is used together with that for the momentum. This independence obviously is fundamental for the usefulness of the whole method. So the Hartree-profiles which have been proved useful before may be taken again. Hartree[4] tabulated the velocity profiles for the case $U\sim s^m$ where the boundary layer equation (2) can be tackled analytically. For each of his profiles (for various m) the integrations for δ_1, δ_2 and δ_3 were carried out numerically and the functions $A, B, \ldots G$ calculated (*see* Table I). It was not necessary to integrate the dissipation D to find $g(H)$ after equation (7), as for these profiles

$$g=-\lambda f \quad \text{with} \quad \lambda=-\frac{\delta_2{}^2}{U}\left(\frac{\partial^2 u}{\partial n^2}\right)_{n=0}. \quad . \quad . \quad . \quad . \quad (14)$$

For, in this case, $U\sim s^m$, the velocity profile does not change its shape at all along the distance from the stagnation point s as long as m is constant. Hence $dH/ds=0$ and, according to equation (6), $g=-(1/U)(dU/ds)Xf$. Further, as these are exact solutions the general boundary condition for equation (2)

$$\lambda=\frac{\delta_2{}^2}{\nu}\frac{dU}{ds}=\frac{X}{U}\frac{dU}{ds}\left(\text{and by definition } \lambda=-\frac{\delta_2{}^2}{U}\left(\frac{\partial^2 u}{\partial n^2}\right)_0\right). \quad . \quad (15)$$

is satisfied, and so simply $g=-\lambda f$. Yet it is just this boundary condition (15) which has been dropped in favour of the energy equation in the approximate method already described. Therefore, it will be fulfilled only approximately when the relation $U\sim s^m$ no longer holds.

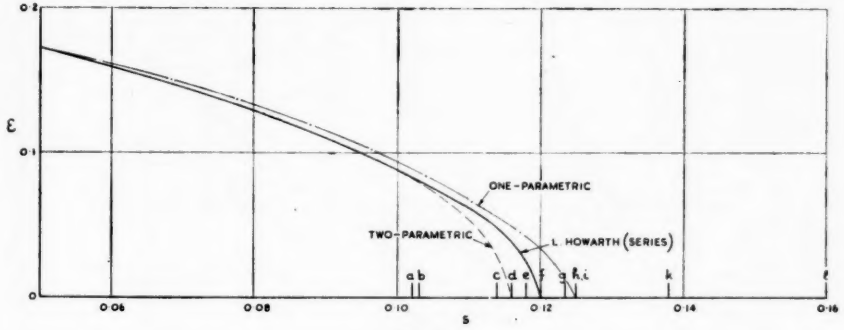

Fig. 1.

Howarth's flow: $U=1-s$. Comparison of the dimensionless skin friction ε calculated by various methods.

3. Examples

To check the accuracy of the calculation method the plane boundary layer has been computed for the case $U=1-s$, which Howarth[5] has treated analytically by series. The calculation begins at $s=0{\cdot}05$ where $\delta_2=0{\cdot}162$, $X=0{\cdot}162^2\,(1-0{\cdot}05)=0{\cdot}0249$ and $H=2{\cdot}77$, $\varepsilon=0{\cdot}172$. The result as given in Fig. 1 (the dimensionless skin friction over the distance s) deviates very little from the exact solution. The separation point is only found by extrapolating as the range of Table I is not great enough. Namely, between $H=3{\cdot}480$ and $4{\cdot}031$ (for the separation profile) no profiles are tabulated by Hartree so that an extension of Table I is impossible without computing more velocity profiles afresh. However, in practice the laminar boundary layer will turn into a turbulent one at some distance before separation. Hence it would not pay to strive after greater accuracy near to the separation point itself.

For comparison Fig. 1 gives the separation point as found by various methods when starting at $s=0{\cdot}05$:

(*a*) Th. v. Kármán-C. B. Millikan[6] $s_{sep}=0{\cdot}102$

(*b*) A. Walz (Hartree profiles; based on the momentum equation only)[7] $=0{\cdot}103$

(*c*) A. Walz (Hartree-profiles)[2] $=0{\cdot}114$

(*d*) Two parametric class of velocity profiles[1] $=0{\cdot}116$

(*e*) H. Görtler (relaxation method)[8] $=0{\cdot}118$

(*f*) L. Howarth (series)[5] $=0{\cdot}120$

(*g*) D. Meksyn (series)[9] $=0{\cdot}1235$

(*h*) Present method $=0{\cdot}125$

(*i*) A. Walz (profile: polynomial of the fourth degree)[2] $=0{\cdot}125$

(*k*) H. Schlichting (profile: polynomial of the sixth degree)[10] $=0{\cdot}138$

(*l*) K. Pohlhausen (profile: polynomial of the fourth degree)[11] $=0{\cdot}160$.

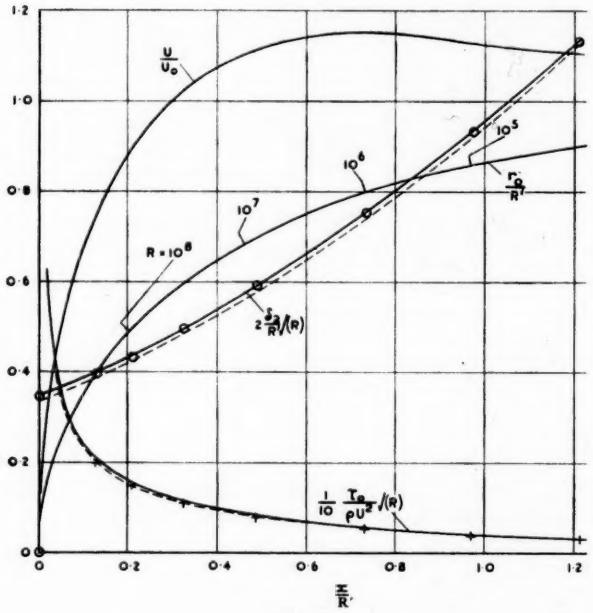

Fig. 2.

Half-body of revolution (produced by a single source): Velocity outside the boundary layer, momentum thickness, skin friction and neutral stability point for various Reynolds numbers.

Further, the boundary layer has been computed for three types of head of an infinite long body of revolution:

(*a*) half-body
(*b*) hemisphere and cylinder
(*c*) $\frac{1}{4}$-calibre rounded head and cylinder.

The pressure distribution in (*b*) and (*c*) was that calculated recently by Vandrey[12] by a new method (applicable to bodies with discontinuous curvature of the meridian). As these pressure distributions are calculated for ideal fluid the results as given in Figs. 2, 3 and 4 hold only for high Reynolds numbers.

The strokes on the surface of the bodies indicate the position of the neutral stability point at the respective Reynolds number, *i.e.* the point after which the laminar layer might become unstable. The actual transition to the turbulent boundary layer will occur between this point and the separation point. The estimation of the neutral stability point is given in the Appendix.

In (*b*), $U(s)$ was given only at a few points s, as indicated by Fig. 3, so that the actual curve is rather uncertain near the end of the head. But obviously as the separation is there in any case it does not matter very much.

Each case needed only a couple of hours' computing work.

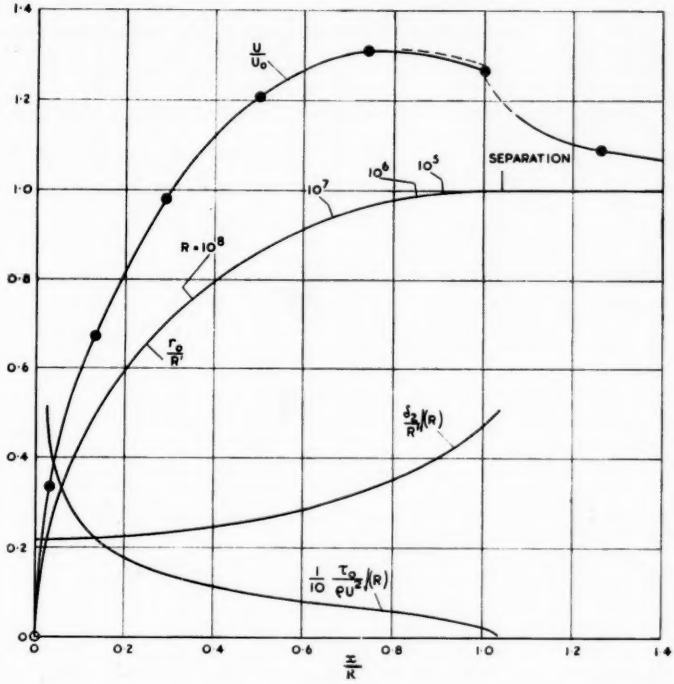

Fig. 3.

Hemispherical head: Velocity outside the boundary layer, momentum thickness, skin friction and neutral stability point for various Reynolds numbers.

4. Comparison with Another Very Simple Method

For plane boundary layers a somewhat simpler calculation method has been proposed by H. Holstein and T. Bohlen[14] and A. Walz[15] in Germany and by Young and Winterbottom[16] and, in detail, by Thwaites[17] in this country. It is based on the momentum equation (5) alone which might be written

$$\frac{dX}{ds} = -(3+2H)\frac{X}{U}\frac{dU}{ds} + 2\varepsilon. \qquad . \quad . \quad . \quad (5a)$$

Supposing that the velocity profiles can be represented by a one parametric class, the right side of (5*a*) is a function of the parameter H alone. For ε this is obvious. Yet it holds also for $(X/U)\,dU/ds$ which equals another parameter

$$\lambda(H) = -\frac{\partial^2 u/U}{\partial\,(n/\delta_2)^2}$$

(compare equation (15)) according to the general boundary condition which has

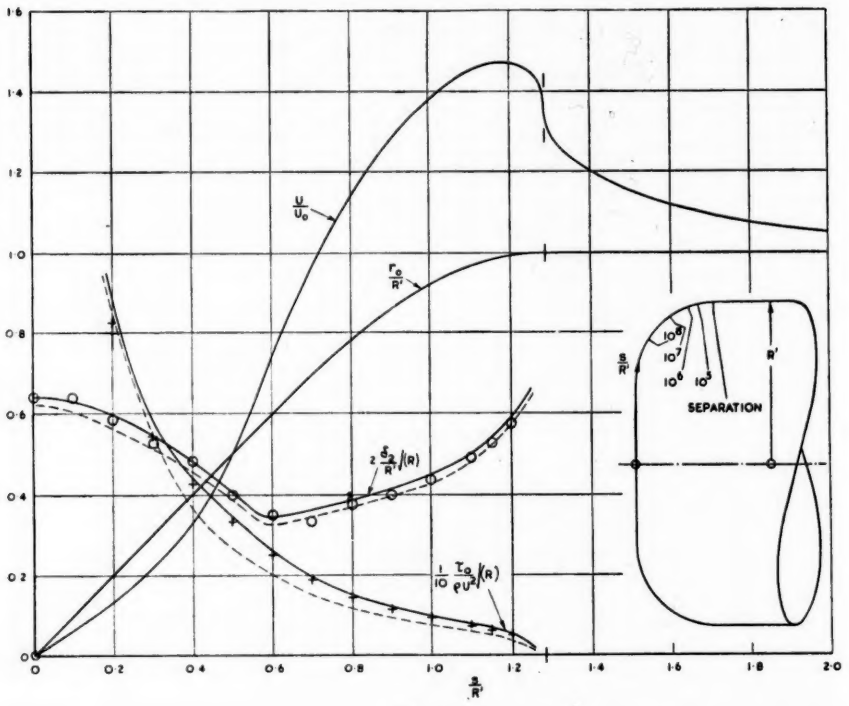

Fig. 4.

1/4-calibre head: Velocity outside the boundary layer, momentum thickness, skin friction and neutral stability point for various Reynolds numbers.

been dropped in the method already described. Hence, the left side of equation (5*a*). dX/ds, is also a function of H alone.

Using Thwaites's notation for $\lambda = -m$, $\varepsilon = l(m)$ and $L(m) = (2H+4)m + 2l$, the momentum equation becomes

$$\frac{dX}{ds} = L(m) - m. \qquad . \quad . \quad . \quad . \quad . \quad (16)$$

For various known solutions of plane boundary layer problems the relation $L(m)$ is almost the same, which proves the main assumption of a one parametric class of velocity profiles. Furthermore, $L(m)$ can be approximated by a linear function such as

$$L(m) = a + bm. \qquad . \quad . \quad . \quad . \quad . \quad (17)$$

Then equation (16) can easily be integrated to give an explicit formula

$$X = U\frac{\delta_2^2}{\nu} = \frac{a}{U^{b-1}}\int_0^s U^{b-1}\,ds, \qquad (18)$$

and the parameter is to be found by

$$m = -\frac{X}{U}\frac{dU}{ds}. \qquad (19)$$

For axi-symmetric flow Mangler's transformation[3] alters this equation (18) into

$$X = \frac{a}{U^{b-1}}\frac{1}{r_0^2}\int_0^s U^{b-1} r_0^2\,ds \qquad (20)$$

with $r_0(s)$ = local radius of the body of revolution. This might also be written

$$\frac{\delta_2^2}{R'^2}R = \frac{a}{(U/U_0)^b}\frac{1}{r_0/R'}\int_0^{s/R'}(U/U_0)^{b-1}(r_0/R')^2\,ds/R', \qquad (21)$$

$$m = -\frac{dU/U_0}{ds/R'}\left(\frac{\delta_2}{R'}\sqrt{R}\right)^2 \qquad (22)$$

and

$$\frac{\tau_0}{\rho U^2}\sqrt{R} = \frac{1}{U/U_0}\frac{R'}{\delta_2\sqrt{R}}\,l(m), \qquad (23)$$

with $l(m)$ after Table I in (17). The results of using this formula are given for two examples in Figs. 2 and 4 (half-body and ¼-calibre rounded head). The broken lines correspond to the constants as proposed by Thwaites[17], $a = 0{\cdot}45$ and $b = 6$.

As he was interested mainly in (plane) cases with increasing pressure—whereas in these examples the region with decreasing pressure is of importance—better results (single marked points in Figs. 2 and 4) are reached by $a = 0{\cdot}45$ and $b = 5{\cdot}5$*.

Thus it is shown that this simple formula, (18) or (20), can be used for axi-symmetric flow as well as for plane flow with an accuracy sufficient for most practical purposes. But it seems advisable to make $b = 5{\cdot}5$ as long as $dU/ds > 0$ and afterwards $b = 6$ for $dU/ds < 0$, with $a = 0{\cdot}45$ in both cases.

APPENDIX

For practical use some known formulae are given, (*a*) for the flow near the stagnation point and, (*b*) to estimate the indifference point.

**Note added in proof*: Meantime E. TRUCKENBRODT, *Ing.-Arch.* 20 (1952), p. 211, proposed the same method with $a = 0{\cdot}47$ and $b = 6{\cdot}0$.

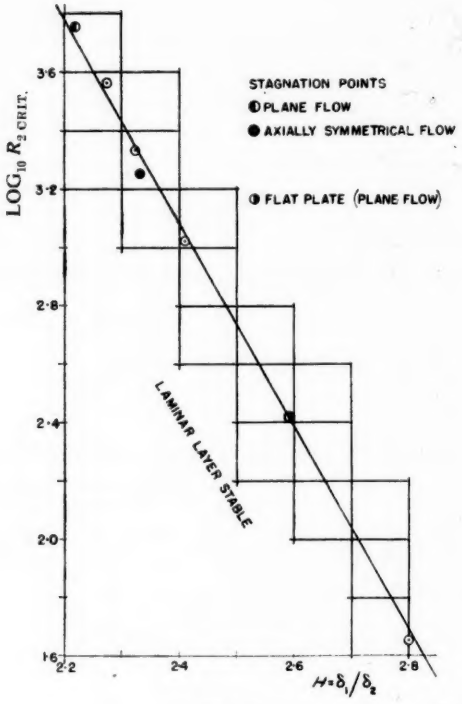

Fig. 5.

Diagram for estimating the neutral stability point from the profile parameter H and the local Reynolds number of the boundary layer (for Hartree profiles).

(*a*) *Flow near the stagnation point*

(α) Two-dimensional:

$$U/U_0 = u_1(s/R'), \quad H = 2{\cdot}21_7, \quad \varepsilon = 0{\cdot}36_0, \quad X = 0{\cdot}0845\,s,$$

$$\frac{\tau_0}{\rho U^2}\sqrt{R} = \frac{1{\cdot}233}{\sqrt{u_1}}\,\frac{1}{s/R'}, \quad \frac{\delta_2}{R'}\sqrt{R} = \frac{0{\cdot}292}{\sqrt{u_1}}.$$

(β) Axi-symmetric:

$$U/U_0 = u_1(s/R'), \quad H = 2{\cdot}29_7, \quad \varepsilon = 0{\cdot}32_5, \quad X = 0{\cdot}0613\,s,$$

$$\frac{\tau_0}{\rho U^2}\sqrt{R} = \frac{1{\cdot}312}{\sqrt{u_1}}\,\frac{1}{s/R'}, \quad \frac{\delta_2}{R'}\sqrt{R} = \frac{0{\cdot}248}{\sqrt{u_1}}.$$

(*b*) *Neutral stability point*

Schlichting[10] and Pretsch[13] have shown that it is possible to attach to each boundary layer profile approximately a critical Reynolds number:

$$R_{2\text{crit}} = (U\,\delta_2/\nu)_{\text{crit}}.$$

If the actual R_2 is smaller, the laminar layer is stable in any case; if $R_2 > R_{2\text{crit}}$ the layer is unstable to certain disturbances so that transition can occur. The real transition to the turbulent layer will take place somewhere between this neutral stability point and the theoretical separation point of the laminar boundary layer.

Usually $R_{2\text{crit}}$ is given as a function of the parameter λ. Yet as λ is determined by H for one parametric classes, Fig. 5 gives directly $R_{2\text{crit}}$—actually the logarithm of it—depending on H.

This diagram holds for axially symmetrical flow as well as for plane flow. It is to be seen that a good approximation is given by the straight line corresponding to

$$(R_2)_{\text{crit}} = e^{26{\cdot}3 - 8H} \quad . \quad . \quad . \quad . \quad . \quad (24)$$

When $X(s)$ and $H(s)$ have been calculated the critical Reynolds number of the main flow $R_{\text{crit}} = (U_0 R'/\nu)_{\text{crit}}$ can be estimated by (24) for any distance s of the neutral stability point from the stagnation point, respectively the position of the neutral stability point s for a given Reynolds number R. For this it is practical to write (24) as follows

$$\log_{10} R_{\text{crit}} = 7{\cdot}55 - 6{\cdot}95\,(H - 2{\cdot}2) - \log_{10}\left(\frac{X}{R'}\,\frac{U}{U_0}\right). \quad . \quad . \quad (25)$$

ACKNOWLEDGMENTS

The author wishes to thank Mr. G. Owen for his assistance with the computing work for Table I. The work described was done in the Admiralty Research Laboratory, Teddington, and it is published on the recommendation of the Aeronautical Research Council and by permission of the Admiralty.

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Buckling of Oblique Plates with Clamped Edges Under Uniform Shear

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Summary: In a previous paper[1] the author derived, by the Rayleigh-Ritz method, a determinantal equation for obtaining the critical magnitude of any uniform system of edge stress applied to a clamped oblique plate. In the present paper this equation is used to derive values for the buckling stresses of clamped oblique plates subjected to pure shear along and perpendicular to two edges of the plate.

As would be expected, it is found that reversal of the direction of the shear produces a change in the critical value. The lower value occurs when the shear is tending to increase the obliquity of the plate and critical values corresponding to this direction are calculated for 45° oblique plates with side ratios varying between 3/5 and 5/3. The critical values obtained for the reverse direction were obviously very inaccurate, due to the inadequacy of the series which was used to represent the buckling mode. As yet no satisfactory method has been found for overcoming this difficulty.

It is shown that for 45° oblique plates with clamped edges the maximum ratio between the larger and smaller critical shear stresses is 2·69, as compared with 8·00 when the edges are simply supported. These are the limiting values for very short plates.

1. Introduction

In a previous paper[1] the author considered the stability of a flat oblique plate (*i.e.* in the shape of a parallelogram) with all edges clamped, subjected to any uniform distribution of edge traction. The Rayleigh-Ritz method was used and a determinantal equation, giving the critical combinations of edge traction, was obtained. This was used to obtain numerical results for the critical stress when the plate was subjected to uniform compression in a direction parallel to two of the sides.

In the present paper the analysis of the previous paper is used to obtain some numerical results for the critical shear stresses of clamped oblique plates, when the shear is applied along and perpendicular to two of the sides of the parallelogram. This means that along these two sides the traction is a pure shear stress, whereas along the other two inclined sides the traction consists of both shear and direct stresses of such magnitude that every infinitesimal rectangular element is in a state of pure shear (Fig. 1). The computation is confined to plates having an angle of obliquity of 45° and side ratios varying between 3/5 and 5/3.

Unlike a rectangular plate, it was found that a reversal of the direction of the shear stress causes a change in its critical value. The lower value occurs when the shear is tending to increase the obliquity of the plate. Numerical values were obtained for the critical shear stresses when applied in this direction, and involved the calculation of the roots of eighth-order determinants.

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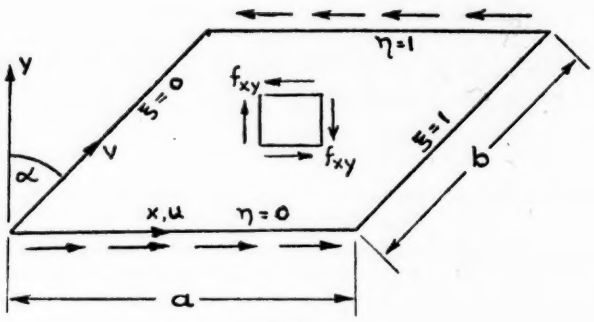

Fig. 1.
Details of oblique plate.

The critical values obtained for the reverse direction were obviously very inaccurate; this could be seen by an inspection of the convergence of the roots as the order of the determinant was increased. In an attempt to discover what order of determinant was necessary to obtain an accurate value, the case of an equal-sided oblique plate was treated more fully. The determinant was extended to twelfth order, but the roots still showed little sign of converging. It was therefore decided that the series of Iguchi functions which was used to express the mode was inadequate unless a very large number of terms was used. Accordingly, an alternative series was tried, but this gave even worse results.

At the moment therefore no satisfactory results for the larger critical shear stresses have been obtained. It is clear, however, that as the length of the plate is increased indefinitely, so that an infinite strip under edge shear is obtained, the two critical shear stresses tend asymptotically to the same value. It is also to be expected that the ratio between the two shear stresses will increase as the plate becomes shorter, the limiting case being an infinitely wide oblique plate. For an obliquity of 45° this case can be readily solved and the results show that, when all edges are clamped, the limiting ratio between the two critical shear stresses is 2·69. The corresponding figure for a plate with all edges simply supported is shown to be 8. It therefore appears that the phenomenon is likely to be more marked in a simply supported plate than a clamped one.

It is shown in Section 3 that on the whole the smaller critical shear stress of an oblique panel is somewhat less than that of an equivalent rectangular panel *of the same weight*. A comparison between oblique and rectangular panels in compression shows exactly the reverse effect.

Notation

a, b lengths of the sides of the parallelogram
α angle of obliquity of the parallelogram
} Fig. 1
t thickness of the plate

D	flexural rigidity of the plate
λ	$=a/b$
x, y	Cartesian co-ordinates (Fig. 1)
u, v	oblique co-ordinates (Fig. 1)
ξ, η	non-dimensional oblique co-ordinates equal to u/a and v/b respectively
X, Y	Cartesian co-ordinates (Fig. 6)
X', Y'	$=X/(a\cos\alpha)$, $Y/(a\cos\alpha)$ respectively
w	lateral deflection of the buckled plate
f_{xy}	critical shear stress (Fig. 1)
β_{xy}	$=f_{xy}b^2t\cos^3\alpha/D$
k	$=f_{xy}\dfrac{(a\cos\alpha)^2t}{\pi^2D}$
ρ	ratio between the larger and smaller critical shear stresses of an oblique plate
a_{pq}	coefficient of the general term in the series representation of w (equation (1))
F_p, G_q	functions defined by equations (3) or (4)
A_{mpnq}	the general element of the stability determinant
$I_{mp}^{(r)}$	definite integrals defined by equations (7)
m, n, p, q	positive integers

2. The Stability Determinant

The plate under consideration is shown in Fig. 1; the angle of obliquity is α and the lengths of the sides are a and b. It is subjected to a pure shear stress, f_{xy}, which is considered positive when in the direction shown. Oblique co-ordinates u and v are used, and the non-dimensional oblique co-ordinates, ξ and η, are defined by the equations

$$\xi=u/a, \qquad \eta=v/b.$$

The equations of the sides of the plate are then $\xi=0, 1$ and $\eta=0, 1$. The boundary conditions are

$$w=\partial w/\partial\xi=0 \quad \text{at} \quad \xi=0,\ 1$$
$$w=\partial w/\partial\eta=0 \quad \text{at} \quad \eta=0,\ 1,$$

where $w(\xi, \eta)$ is the lateral deflection of the plate when buckling occurs.

A suitable expression for w is

$$w=\sum_{p=1}^{\infty}\sum_{q=1}^{\infty} a_{pq}F_p(\xi)G_q(\eta) \quad . \quad . \quad . \quad . \quad (1)$$

where, in order to satisfy the boundary conditions term by term, the set of functions F and G are such that

$$\left.\begin{aligned} F_p(0) &= F_p(1) = F_p'(0) = F_p'(1) = 0 \\ G_q(0) &= G_q(1) = G_q'(0) = G_q'(1) = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (2)$$

A possible form for the F and G functions is that first used by Iguchi[2],

$$\left.\begin{aligned} F_p(\xi) &= \xi(\xi-1)^2 + (-1)^p \xi^2 (\xi-1) - \frac{1}{p\pi} \sin p\pi\xi \\ G_q(\eta) &= \eta(\eta-1)^2 + (-1)^q \eta^2 (\eta-1) - \frac{1}{q\pi} \sin q\pi\eta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (3)$$

These were used by the author[1] for determining the critical compressive stresses of oblique plates.

Alternatively we may take

$$\left.\begin{aligned} F_p(\xi) &= \sin \pi\xi \sin p\pi\xi \\ G_q(\eta) &= \sin \pi\eta \sin q\pi\eta \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

These functions lead to a somewhat simpler stability determinant by virtue of their near orthogonality. It will be seen later, however, that for a given order of determinant they give less accurate critical stresses. This was also found to be true for the case of an oblique plate under uniform compression.

The condition for neutral stability of the buckled plate now leads to the determinantal equation[1].

$$\begin{array}{c} \downarrow \quad \longrightarrow p,q \\ m,n \left| A_{mpnq} \right| \end{array} = 0, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where

$$A_{mpnq} = \frac{1}{\lambda^2} I_{mp}^{(4)} I_{nq}^{(0)} + \lambda^2 I_{mp}^{(0)} I_{nq}^{(4)} + 2(1+2\sin^2\alpha) I_{mp}^{(2)} I_{nq}^{(2)} - 4\sin\alpha \left(\frac{1}{\lambda} I_{mp}^{(3)} I_{nq}^{(1)} + \right.$$

$$\left. + \lambda I_{mp}^{(1)} I_{nq}^{(3)} \right) + 2\beta_{xy} (\lambda I_{mp}^{(1)} I_{nq}^{(1)} - \sin\alpha I_{mp}^{(2)} I_{nq}^{(0)}) \quad . \quad . \quad (6)$$

and

$$I_{mp}^{(r)} = \int_0^1 F_m(\xi) \frac{d^r}{d\xi^r} F_p(\xi)\, d\xi = \int_0^1 G_m(\eta) \frac{d^r}{d\eta^r} G_p(\eta)\, d\eta \quad . \quad (7)$$

The constant λ is the side ratio a/b, while β_{xy} is the buckling stress coefficient defined by the equation

$$\beta_{xy} = f_{xy} \frac{b^2 t \cos^3\alpha}{D} \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

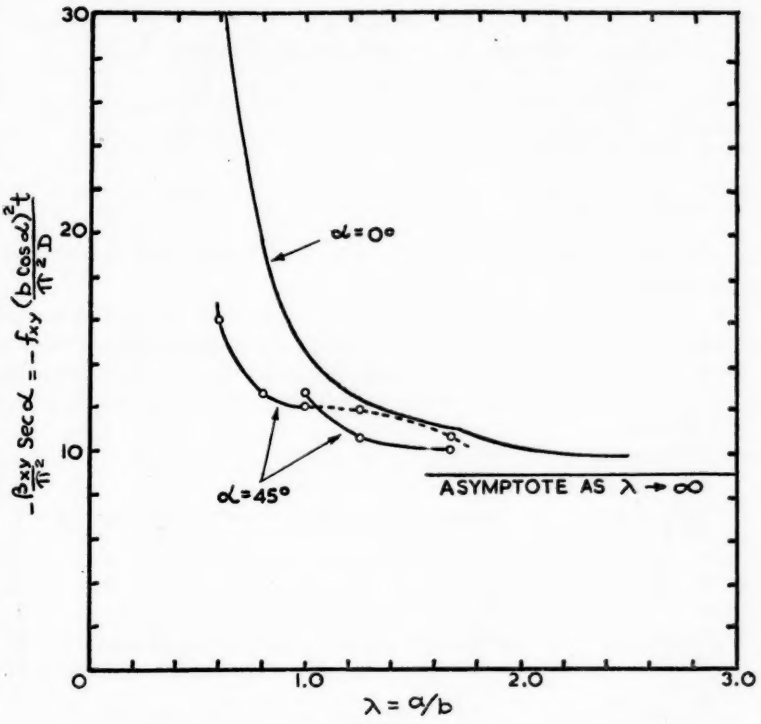

Fig. 2.
Curve showing the smaller critical shear stress of 45° clamped oblique plates.

The buckling mode was shown[1] to be either symmetrical or anti-symmetrical about the centre of the plate. For a symmetrical buckle we include in the determinant only those elements in which both $(m+n)$ and $(p+q)$ are even, while for an anti-symmetrical buckle we take $(m+n)$ and $(p+q)$ to be odd. The governing mode is that which gives the lowest critical stress.

For given values of λ and α, every element of the determinant in equation (5) contains a constant term and one proportional to the coefficient β_{xy}. The numerically smallest positive and negative values of β_{xy} that make the determinant zero give the required critical stresses. In all cases it appeared that the negative critical stress is numerically smaller than the positive one.

3. The Smaller Critical Stress

The numerically lowest negative roots of the stability determinants were calculated by a process of trial and error, as described in Ref. 1. The results for 45° oblique plates, as obtained from eighth-order determinants, are given in Table I. The rows and columns included were those corresponding to m, n, p

TABLE I

CRITICAL STRESSES FOR CLAMPED OBLIQUE PLATES. $\alpha = 45°$.

	$\lambda = a/b$	0·6	0·8	1·0	1·25	1·667
$\frac{\beta_{xy}}{\pi^2}$	Symmetrical buckle. $(m+n)$ and $(p+q)$ even	−11·37	−8·99	−8·60	−8·46	−7·61
	Anti-symmetrical buckle. $(m+n)$ and $(p+q)$ odd	—	—	−8·96	−7·51	−7·16

and $q \leqslant 4$, and the Iguchi functions (equations (3)) were used to represent the buckling mode.

It will be seen from Fig. 1 that a negative shear stress is one tending to increase the obliquity of the plate. The results of Table I are shown in Fig. 2, which gives the curve of $(\beta_{xy}/\pi^2)\sec\alpha$ plotted against the side ratio λ. This form for the buckling stress coefficient is chosen for the following reason. It will be seen from equation (8) that

$$\frac{\beta_{xy}}{\pi^2}\sec\alpha = f_{xy}\frac{(b\cos\alpha)^2 t}{\pi^2 D}.$$

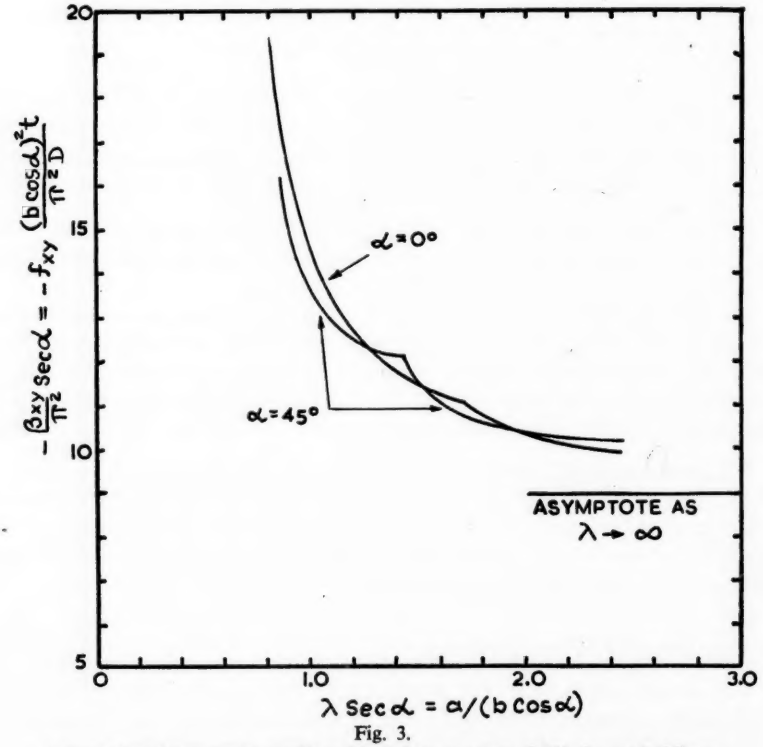

Fig. 3.

Comparison between the smaller critical shear stresses of 45° clamped oblique plates with those of rectangular plates of equal weight.

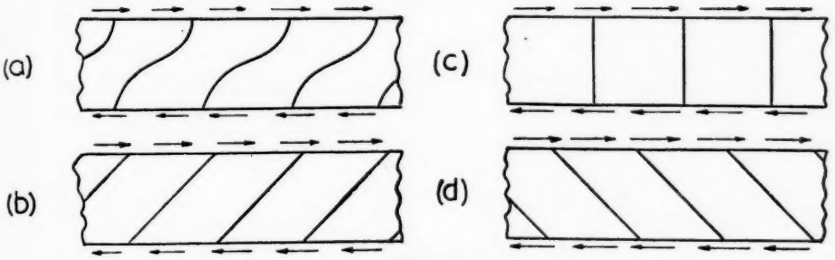

Fig. 4.
Infinite strips under shear.

Now $b\cos\alpha$ is the width of the plate in a direction perpendicular to the sides of length a. Hence for a very long plate ($\lambda \longrightarrow \infty$) the value of $(\beta_{xy}/\pi^2)\sec\alpha$ becomes independent of α and equal to the value 8·98 for an infinite strip with clamped edges, of width $b\cos\alpha$, under uniform edge shear[3]. The curve for $\alpha=45°$ is therefore asymptotic (for large values of λ) to the curve[4] for $\alpha=0$ which is also shown in Fig. 2.

The volume of an oblique plate is equal to $tab\cos\alpha$. Thus, in order to make a direct comparison between the critical shear stresses of oblique and rectangular panels *of equal weight* the buckling stress coefficient $(\beta_{xy}/\pi^2)\sec\alpha$ should be plotted against the ratio $a/(b\cos\alpha)=\lambda\sec\alpha$. This is shown in Fig. 3 and it will be seen that on the whole the 45° oblique plate has a somewhat *smaller* buckling stress than the equivalent rectangular plate.

This result might have been anticipated in view of the fact that the buckling mode of an infinite strip in shear contains nodal lines similar to those shown sketched in Fig. 4(*a*). If the strip is subdivided into oblique panels by a series of equidistant stiffeners in the same general direction as the natural nodes of the infinite strip (Fig. 4(*b*)) it might be expected that the critical shear stress would be less than if the subdivision were into rectangular panels (Fig. 4(*c*)). In a similar way it might be expected that for a strip subdivided into oblique panels as in Fig. 4(*d*) the critical shear stress would be higher than for the rectangular panels, although the results obtained for this case are not sufficiently accurate to confirm this conclusively (Section 4).

It is worthy of note that if an infinite strip is subjected to compression parallel to the edges the natural nodal lines are straight and perpendicular to the edges. Thus we might expect that in compression an oblique plate would have a *greater* buckling stress than the equivalent rectangular plate. This can be seen from the results of Ref. 1 for clamped oblique plates, which are shown plotted in Fig. 5. The work of Anderson[6] on the compressive buckling of continuous flat sheet sub-divided into oblique panels gives further justification for this conclusion.

It is therefore concluded that, although obliquity gives an increase in the critical compressive stress of a panel of given weight, it also causes a decrease on the whole of the smaller critical shear stress.

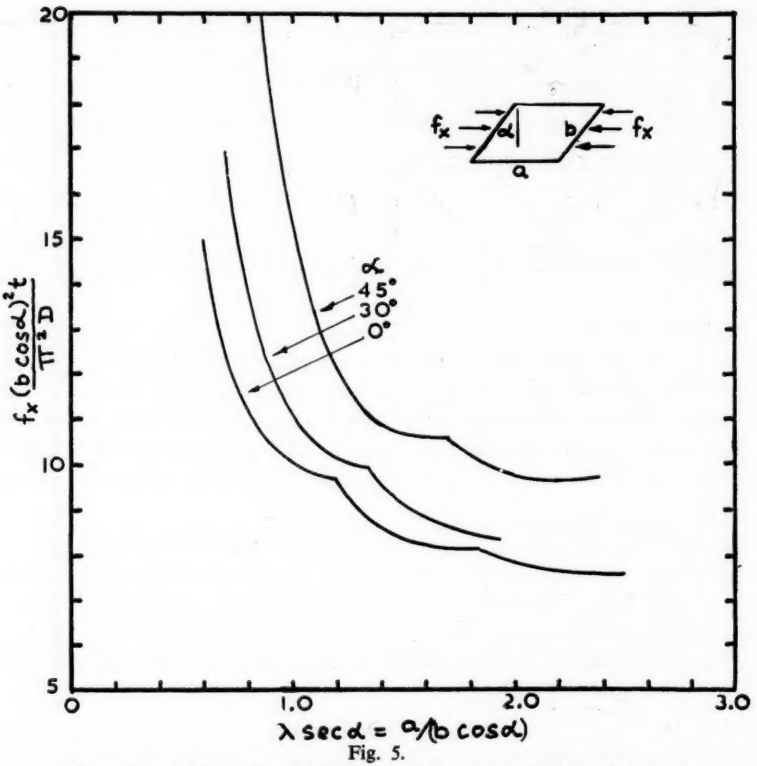

Fig. 5.
Comparison between the critical compressive stresses of clamped oblique and rectangular plates of equal weight.

4. The Larger Critical Stress

The lowest positive root of the stability determinant for the symmetrical buckling of an equal-sided panel of 45° obliquity ($\lambda=1$, $\alpha=45°$) was calculated in exactly the same way as the negative root. The Iguchi functions, given by equations (3), were again used to express the mode, but this time the roots of successive determinants up to the eighth order showed no sign of convergence, *i.e.* each unit increase in the order of the determinant gave a substantial decrease in the root. In an attempt to discover what order is required to give an accurate value, the determinant was extended to the twelfth order* and the root again calculated. The results for the determinants of ninth to twelfth orders are given in Table II.

It will be seen that there is still no sign of convergence of the successive roots, and it was therefore concluded that a very large number of terms in the series for w is required to represent the mode adequately.

*This corresponded to the addition of terms in the series, equation (1), involving the coefficients a_{51}, a_{15}, a_{53} and a_{35}.

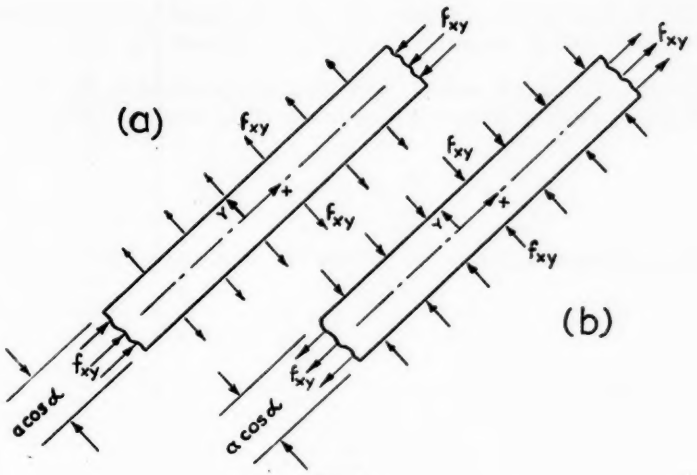

Fig. 6.
Limit of a 45° oblique plate under shear as $\lambda \to 0$.

An alternative form, given by equations (4), was then used for the F and G functions and the roots again calculated. It was found, however, that the accuracy was worse than that obtained by the use of the Iguchi functions.

Thus, at present, no satisfactory results have been obtained for the larger critical stress.

5. The Limiting Values of the Ratio between the Larger and Smaller Critical Stresses

It is obvious that, as $\lambda \longrightarrow \infty$, so that an infinite strip under edge shear is obtained, the direction of shear becomes immaterial. Hence if we define ρ as the ratio between the numerical values of the larger and smaller critical stresses we have

$$\rho \longrightarrow 1 \quad \text{as} \quad \lambda \longrightarrow \infty.$$

This result is true whatever the boundary conditions of the panel.

It is also to be expected that the ratio ρ will increase as the side ratio λ decreases, and that ρ will have its greatest value as $\lambda \longrightarrow 0$; we then arrive in effect at an infinite strip of width $a\cos\alpha$. Considering the case of an oblique plate in which $\alpha = 45°$, it is evident that if f_{xy} is *positive* this strip is subjected to a longitudinal *compressive* stress f_{xy} and a lateral *tensile* stress also equal to f_{xy}, as shown in

TABLE II

Order of determinant	9	10	11	12
β_{xy}/π^2	17·70	17·41	16·29	13·03

Fig. 6(*a*). If, however, f_{xy} is *negative* the strip is subjected to a longitudinal *tensile* stress f_{xy} and a lateral *compressive* stress f_{xy}, as shown in Fig. 6(*b*).

Consider first the case where f_{xy} is positive, as represented by Fig. 6(*a*). Taking axes X and Y, where X is along the centre line of the strip and writing

$$X' = \frac{X}{a\cos\alpha}, \qquad Y' = \frac{Y}{a\cos\alpha},$$

the differential equation for the lateral deflection w when buckling occurs is

$$\left(\frac{\partial^2}{\partial X'^2} + \frac{\partial^2}{\partial Y'^2}\right)^2 w = -\pi^2 k\left(\frac{\partial^2 w}{\partial X'^2} - \frac{\partial^2 w}{\partial Y'^2}\right), \quad . \quad . \quad (9)$$

where k is defined by the equation

$$k = f_{xy}\frac{(a\cos\alpha)^2 t}{\pi^2 D}. \quad . \quad . \quad . \quad (10)$$

On using equations (8) and (10) we see that

$$\frac{\beta_{xy}}{\pi^2}\sec\alpha = \frac{k}{\lambda^2}. \quad . \quad . \quad . \quad . \quad (11)$$

The buckle will be sinusoidal in the X direction and accordingly we write

$$w = W(Y')\sin(\Omega\pi X'), \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where Ω is a constant defining the wavelength of the buckle in the X direction.

Substitution of equation (12) into equation (9) yields an ordinary differential equation in W whose general solution is

$$W = A\cosh 2\theta Y' + B\sinh 2\theta Y' + C\cos 2\phi Y' + D\sin 2\phi Y' \quad . \quad (13)$$

where
$$\theta^2 = \frac{\pi^2}{8}\left[(k^2 + 8\Omega^2 k)^{\frac{1}{2}} + (k + 2\Omega^2)\right]$$

and
$$\phi^2 = \frac{\pi^2}{8}\left[(k^2 + 8\Omega^2 k)^{\frac{1}{2}} - (k + 2\Omega^2)\right] \quad . \quad . \quad . \quad (14)$$

Suppose now that the edges $\xi = 0$ and 1 of the oblique plate are clamped*, *i.e.* both edges of the infinite strip are clamped. The boundary conditions for W are then

$$W = dW/dY' = 0 \quad \text{at} \quad Y' = \pm\tfrac{1}{2}. \quad . \quad . \quad . \quad . \quad (15)$$

*Note that no mention is made of the conditions along the other two edges of the oblique plate. These are immaterial when $\lambda \to 0$.

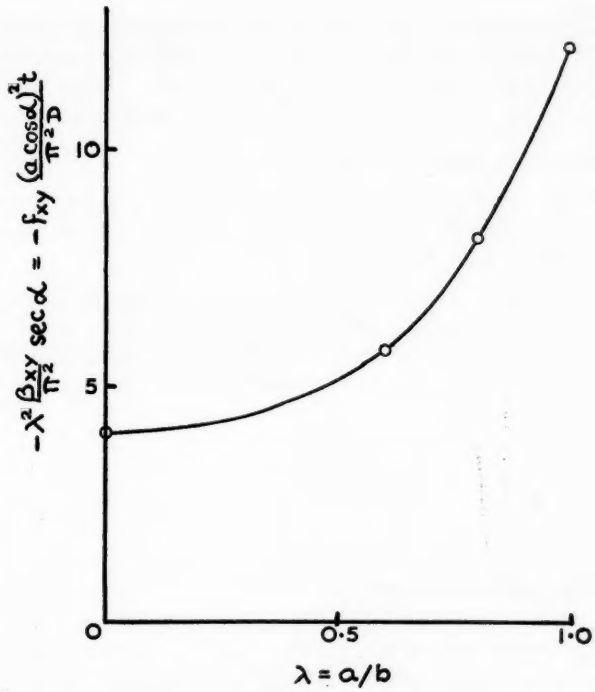

Fig. 7.
Curve giving the smaller critical shear stress of short 45° clamped oblique plates.

Also the lowest buckling stress will be associated with a mode symmetrical about $Y=0$ so that $B=D=0$. Equations (13) and (15) then lead to the transcendental equation

$$\theta \tanh\theta + \phi \tan\phi = 0. \quad . \quad . \quad . \quad . \quad . \quad (16)$$

It will be seen from equations (14) that equation (16) defines k as a function of Ω. It is now a matter of choosing Ω so that k has its minimum value. By trial and error it was found that

$$k = 10{\cdot}76 \quad . \quad . \quad . \quad . \quad . \quad (17)$$

occurring at a wavelength defined by $\Omega = 2{\cdot}05$ approximately.

If on the other hand the edges $\xi = 0$ and 1 of the oblique plate are simply supported, the boundary conditions for W become

$$W = d^2W/dY'^2 = 0 \quad \text{at} \quad Y' = \pm\tfrac{1}{2}. \quad . \quad . \quad . \quad (18)$$

Again the governing mode will be symmetrical, so that $B=D=0$, and equations (13) and (18) now give the equation

$$\cos\phi = 0.$$

It follows therefore that $\phi=\pi/2$, and on using the second of equations (14) we find that

$$k=\frac{(\Omega^2+1)^2}{\Omega^2-1}.$$

On minimising k with respect to Ω we obtain

$$k=8 \qquad . \quad . \quad . \quad . \quad . \quad (19)$$

occurring at a wavelength defined by $\Omega=\sqrt{3}$.

Consider now the case where f_{xy} is negative, as represented by Fig. 6(*b*). It is obvious that the lowest buckling stress corresponds to a mode which is cylindrical (*i.e.* w is independent of X), in which case the longitudinal tensile stress has no effect. Thus the value of the lateral compressive stress f_{xy} required to cause buckling is given by

$$k=4 \qquad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

if the edges of the strip are clamped, and by

$$k=1 \qquad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

if the edges are simply supported.

The results of this analysis are summarised in Table III. This shows the limiting values of $(\beta_{xy}/\pi^2)\sec\alpha$, as given by equations (11), (17), (19), (20) and (21), corresponding to the larger (positive) and smaller (negative) critical stresses of a 45° oblique plate, and also the ratio ρ between the two stresses, as the side ratio λ tends to zero.

The two values of ρ given in Table III indicate that for an oblique plate of given dimensions there is a considerably greater difference between the two critical shear stresses if the edges are simply supported than if they are clamped.

The results of this analysis also enable the results for the smaller critical stress to be extended to values of λ less than those covered by Table I. This may be

TABLE III

CRITICAL SHEAR STRESSES FOR A 45° OBLIQUE PLATE AS $\lambda \longrightarrow 0$

Boundary conditions		*Edges* $\xi=0$ and 1 *clamped*	*Edges* $\xi=0$ *and* 1 *simply supported*
$\frac{\beta_{xy}}{\pi^2}\sec\alpha$	Larger critical stress	$\frac{10{\cdot}76}{\lambda^2}$	$\frac{8}{\lambda^2}$
	Smaller critical stress	$-\frac{4}{\lambda^2}$	$-\frac{1}{\lambda^2}$
ρ		2·69	8·00

done by plotting $\lambda^2(\beta_{xy}/\pi^2)\sec\alpha\,(=k)$ against λ, as shown in Fig. 7. The value for $\lambda=0$ was taken from Table III, while the values for $\lambda=0{\cdot}6$, $0{\cdot}8$ and $1{\cdot}0$ were calculated from Table I. The values for λ lying between 0 and $0{\cdot}6$ may easily be interpolated from the curve.

ACKNOWLEDGMENTS

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Theory of an Accelerated Slender Supersonic Aerofoil

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Summary: The problem of a uniformly accelerated slender supersonic aerofoil is solved in this paper by a perturbation method for the cases when the acceleration is small. Full account is taken of the effect of the acceleration on the form of the shock line.

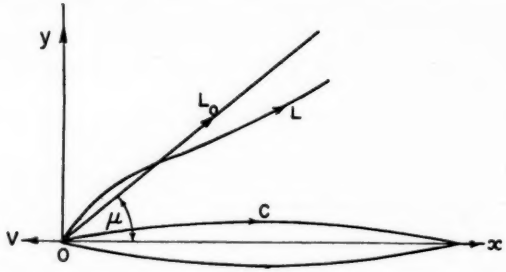

Fig. 1.

Notation

Ox, Oy	axes of reference, O being the nose of the aerofoil and Ox its axis
$\mathbf{k}$	unit vector perpendicular to Ox and Oy, making a right-handed set with the directions of Ox, Oy, respectively
t	time variable
p_0, ρ_0, a_0	constant pressure, density and speed of sound, respectively, upstream of the shock
$\mathbf{u}(=(u,v)), p, \rho$	perturbation velocity of flow, pressure and density, respectively, downstream of the shock
V	velocity of the aerofoil, its direction being opposite to Ox
$\varepsilon(=\dot{V})$	uniform acceleration of the aerofoil
$\xi \equiv \xi(t)$	distance travelled by the aerofoil at time t
μ	given by $a_0 = V \sin\mu$
C	upper side of the aerofoil profile
$F(x)$	given by $y = F(x)$ as the equation for the form of C
L	shock curve
L_0	the line $y = x \tan\mu$.

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Theory

The boundary conditions at the shock[1] are given by

$$u : v : p : \rho = -a_0 \sin\mu : a_0 \cos\mu : \rho_0 a_0^2 : \rho_0 \quad \text{on} \quad L_0; \qquad \text{(A)}$$

and the boundary condition at the aerofoil is

$$v = VF'(x) \quad \text{on} \quad y=0. \qquad \text{(B)}$$

The linearised equations of flow downstream of the shock are

$$\frac{\partial \rho}{\partial t} + V\frac{\partial \rho}{\partial x} + \rho_0 \operatorname{div}\mathbf{u} = 0, \qquad (1)$$

$$\mathbf{grad}\, p + \rho_0\left(\frac{\partial \mathbf{u}}{\partial t} + V\frac{\partial \mathbf{u}}{\partial x}\right) = 0, \qquad (2)$$

and

$$\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)(p - a_0^2\rho) = 0. \qquad (3)$$

Hence, by equations (1) and (3),

$$\frac{\partial p}{\partial t} + V\frac{\partial p}{\partial x} + \rho_0 a_0^2 \operatorname{div}\mathbf{u} = 0. \qquad (4)$$

Moreover, by equation (2), p and $\mathbf{u}$ may be expressed in the forms

$$p = -\rho_0\left(\frac{\partial H}{\partial t} + V\frac{\partial H}{\partial x}\right), \qquad (5)$$

and

$$\mathbf{u} = \mathbf{grad}\, H + \mathbf{curl}\left[\alpha\left(x - \xi(t), y\right)\mathbf{k}\right], \qquad (6)$$

respectively, and therefore, by equation (4),

$$a_0^2\nabla^2 H = \left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right)^2 H. \qquad (7)$$

Now we know the solution of this equation, subject to the boundary conditions (A) and (B), when V is constant[1], and this suggests letting V replace t as an independent variable, so that $\partial/\partial t = \varepsilon\,\partial/\partial V$; whereupon, neglecting ε^2, equation (7) takes the form

$$a_0^2\frac{\partial^2 H}{\partial y^2} - a_0^2\cot^2\mu\frac{\partial^2 H}{\partial x^2} = \varepsilon\left(\frac{\partial H}{\partial x} + 2V\frac{\partial^2 H}{\partial x \partial V}\right). \qquad (8)$$

If, further, we let $H = H_1 + \varepsilon H_2$, where H_1 is the solution when V is constant, then

$$H_1 = -VF(x - y\cot\mu)\tan\mu, \qquad (9)$$

and H_2 is given by the equation

$$a_0^2 \frac{\partial^2 H_2}{\partial y^2} - a_0^2 \cot^2 \mu \frac{\partial^2 H_2}{\partial x^2} = \frac{\partial H_1}{\partial x} + 2V \frac{\partial^2 H_1}{\partial x \partial V}. \quad . \quad . \quad (10)$$

Thus, from equations (9) and (10), it may be seen that

$$\sin^2 \mu \frac{\partial^2 H_2}{\partial y^2} - \cos^2 \mu \frac{\partial^2 H_2}{\partial x^2} = V^{-1} [2 \tan^3 \mu - \tan \mu F'(x - y \cot \mu) + 2 y \sec^2 \mu F''(x - y \cot \mu)], \quad . \quad . \quad (11)$$

the general solution of which is

$$H_2 = -\tfrac{1}{4} V^{-1} \sec^2 \mu \tan^3 \mu (x + y \cot \mu) F(x - y \cot \mu) - \tfrac{1}{2} V^{-1} \sec^4 \mu \cot \mu y^2 F'(x - y \cot \mu) + A(x - y \cot \mu) + B(x + y \cot \mu); \quad . \quad (12)$$

and hence

$$H = -VF(x - y \cot \mu) \tan \mu + \varepsilon [A(x - y \cot \mu) + B(x + y \cot \mu) - \tfrac{1}{4} V^{-1} \sec^2 \mu \tan^3 \mu (x + y \cot \mu) F(x - y \cot \mu) - \tfrac{1}{2} V^{-1} \sec^4 \mu \cot \mu y^2 F'(x - y \cot \mu)]. \quad . \quad (13)$$

The function α, introduced in equation (6), must be of the order of ε, since it is zero when V is constant; and since none of the other functions concerned involve t, save in V and μ, it follows that α must be a function of y only, so that

$$\alpha = \varepsilon \alpha_1 (y). \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

We have now three arbitrary functions, A, B and α_1, which we now determine from the boundary conditions (A) and (B), the result being that

$$\left.\begin{aligned} \alpha_1'(y) &= V^{-1} \sec^2 \mu F'(0) y, \\ B'(x + y \cot \mu) &= \tfrac{1}{4} V^{-1} \sec^2 \mu \tan^3 \mu F'(0)(x + y \cot \mu), \\ \text{and} \quad A'(x - y \cot \mu) &= \tfrac{1}{4} V^{-1} \sec^2 \mu \tan^3 \mu [3(x - y \cot \mu) F'(x - y \cot \mu) - \\ &\quad - 3F(x - y \cot \mu) + F'(0)(x - y \cot \mu)]. \end{aligned}\right\} \quad . \quad (C)$$

Thus, H and α are given by the expressions

$$H = -VF(x - y \cot \mu) \tan \mu + \tfrac{1}{4} \varepsilon V^{-1} \sec^2 \mu \tan^3 \mu [(x^2 + y^2 \cot^2 \mu) F'(0) - 6 y \cot \mu F(x - y \cot \mu) - 6 \int_0^{(x - y \cot \mu)} F(t)\, dt - 2 y^2 \sec^2 \mu \cot^4 \mu F'(x - y \cot \mu)],$$

and $$\alpha = \tfrac{1}{2} \varepsilon V^{-1} y^2 \sec^2 \mu F'(0).$$

This, then, completely determines the solution, from which it is seen that waves are in general propagated along both sets of characteristics, owing to the acceleration, and that vorticity is produced.

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The Delta Wing in a Non-Uniform Supersonic Stream*

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Summary: A generalised conical field theory is developed and is applied to delta wings in a non-uniform stream. It is shown that a non-uniform stream may be characterised by the downwash at all points in space. The lift of a delta wing is found when the downwash in the wing plane is given as a power series in the co-ordinates in the wing plane. The basis of the conical field theory is described in some detail but the results only of the calculation of the lift distribution for various downwashes are given. The solutions of certain integral equations, required in the calculations, are given in the Appendix.

1. Introduction

The co-ordinate system is shown in Fig. 1. The apex of the wing is taken as origin, the x-axis is drawn in the direction of the free stream (the wing is placed symmetrically with respect to this direction), the y-axis is drawn in the starboard direction and the z-axis completes a right-handed co-ordinate system.

It is shown in Section 2 how non-uniform flows arise in practice and it is also shown that the consideration of such flows reduces to the following problem:— Given the downwash w over the wing area, assumed in the $z=0$ plane, as a power series in x and y, determine the x-component of velocity; from this the lift of the wing can be obtained. Linearised supersonic flow theory is used throughout.

There are two types of wing to consider: (*a*) delta wings with *subsonic* leading edges or, in other words, those wings which are swept *inside* the Mach cone from the apex; (*b*) delta wings with *supersonic* leading edges, that is wings for which the leading edges lie *outside* the Mach cone from the apex.

Type (*a*) are the only wings considered here, because the method of sources and sinks can be applied to determine the lift of wings with supersonic leading edges. To solve the problem for wings with subsonic leading edges, conical flow theory is used. The theory of higher order conical flows[2, 7, 11, 13] is used, since the well-known first order theory is not adequate to deal with the present problem. (An exact definition of higher order conical flows is deferred until Section 3.) In this paper the general procedure only is described and the final results are listed in Section 10. The reader is referred to Ref. 6 for further details of the working in any particular case.

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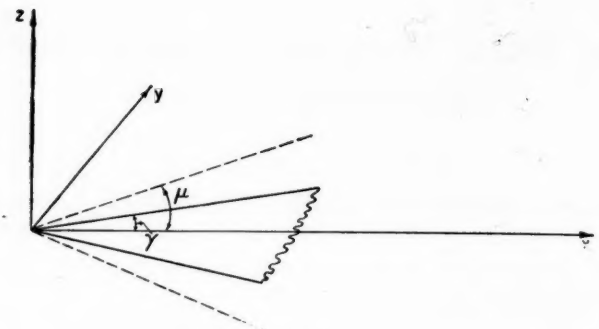

Fig. 1.
The delta wing and associated notation.
Mach lines are denoted by — — — — —
2γ is the angle between the leading edges of the wing
$\cot\gamma = k$, and $\cot\mu = B$.

Notation

a	parameter, less than 1, related to the angle of the delta wing by $a = B/k$
c_r	constants used in the results of the calculation of the lift distribution
k	related to γ by $\cot\gamma = k$
$l(x, y)$	non-dimensional lift distribution of the delta wing
n	"order" of the conical flow
u, v, w	components of the perturbation velocity given by $u = \partial\phi/\partial x$, $v = \partial\phi/\partial y$, $w = \partial\phi/\partial z$
u', u'', v', etc.	defined by $u' = \partial\phi'/\partial x$, $u'' = \partial\phi''/\partial x$, $v' = \partial\phi'/\partial y$ and so on
$u_1(\eta)$	value of $u(\eta, \zeta)$ on real axis (*see* equation (24))
w_{rs}	constant which is the coefficient of $x^r y^s$ in the expression for w' (*see* equation (3))
x, y, z	rectangular Cartesian co-ordinates (Fig. 1)
B	defined by $B^2 = M^2 - 1$
D_1, D_2	constants given in Section 10
E	complete elliptic integral of the second kind, modulus $(1 - a^2)^{\frac{1}{2}}$
K	complete elliptic integral of the first kind, modulus $(1 - a^2)^{\frac{1}{2}}$
M	Mach number
U	mean free stream velocity (Section 2)
U, V, W	analytic functions of complex variable Θ or σ, real parts of which are $u(\eta, \zeta)$, $v(\eta, \zeta)$ and $w(\eta, \zeta)$
X	denotes the quantity $(x^2 - k^2 y^2)^{\frac{1}{2}}$

γ	half the angle between the leading edges of the wing
η, ζ	real and imaginary parts of σ
μ	Mach angle
ξ, θ	co-ordinates defined by (6)
σ	complex variable defined by $\sigma = \eta + i\zeta$
ϕ	perturbation velocity potential
ϕ', ϕ'', ϕ'''	velocity potentials defined in Section 2 and satisfying equation (2)
Θ	complex variable related to σ by (21)
$\overline{\Theta}$	complex conjugate of Θ
Φ_{pqr}	analytic function of σ, with $p+q+r=n$
$\oint$	symbol denoting "principal value integral"
$\mathscr{R}$	"the real part of"

2. Non-uniform Flows

A non-uniform flow is one in which the free stream velocity is a function of the space co-ordinates. It is assumed that the perturbation velocity potential and velocity components are independent of time.

Suppose now that we have an aeroplane flying with velocity U into still air, then it is simpler to consider the aeroplane stationary with a wind whose speed is U blowing past it. The free stream flow past the mainplane can be considered as uniform, if the presence of propellers and so on is neglected; the lift on aerofoils in such a flow is well known. However, this mainplane will certainly disturb the free stream and on reaching the tailplane the wind will be non-uniform. The case of a rectangular tailplane, with large aspect ratio, has been solved by Mirels[10] and we are here considering a delta shaped tailplane. The non-uniformity is best described by the downwash, which is produced by the mainplane, at the surface of the tailplane and which would be present if the tailplane were absent. The methods of calculating the downwash behind a supersonic wing have been given by Robinson[14], Robinson and Hunter-Tod[15] and Ward[18]. A non-uniform flow also occurs in a wind tunnel which is imperfect, in that the liners do not succeed in producing a uniform flow. In a perfect wind tunnel there will be no downwash, so again the non-uniformities in the stream may be described by stating the downwash at all points in the field. The results obtained in the present paper can be used to find the lift of swept wings in supersonic flow when aeroelastic effects are taken into account. A few further results are required which can be derived by methods similar to those used here.

The fundamental linearised equation for the velocity potential, ϕ, is

$$\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = B^2 \frac{\partial^2\phi}{\partial x^2}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where $B^2 = M^2 - 1$, and M is the Mach number.

Since (1) is a linear partial differential equation, the ϕ may be split up into various parts and if each part is a solution of the equation so is their sum. We write

$$\phi=\phi'+\phi''+\phi''' \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and define the three components in the following way:—

(*i*) ϕ' is the perturbation velocity of the flow, due to the upstream wing, when the tailplane is assumed to be absent.

(*ii*) ϕ'' arises out of a cancellation of $w'=\partial\phi'/\partial z$ (the perturbation velocity of the non-uniform stream in the z-direction) at each point of the tailplane plan form area. In other words ϕ'' is to be constructed such that $\partial\phi''/\partial z=-\partial\phi'/\partial z$ for all points in the $z=0$ plane which lie within the tailplane plan form area.

(*iii*) ϕ''' is the velocity potential due to the tailplane, at given angle of incidence, in a uniform stream whose velocity is U.

When we consider the downwash at all points of the tailplane surface due to ϕ', ϕ'' and ϕ''' it is clear that the boundary condition—the flow is tangential to the tailplane—is satisfied. The values of ϕ' and ϕ''' will be assumed known as these have been given elsewhere[3, 12, 14, 18]. We shall assume that w' is known and expanded in a series of the form

$$w'=w_{00}+w_{10}x+w_{01}y+w_{20}x^2+w_{11}xy+w_{02}y^2+\ldots. \quad . \quad . \quad (3)$$

where the w_{rs} are constant coefficients.

The problem thus reduces to finding the value of $u''=\partial\phi''/\partial x$ at all points of the tailplane, given the value of w'', where $w''=-w'$ and w' is given by (3). Having found u'', this will immediately give the additional lift produced by the non-uniformities. It might be thought that it is sufficient to treat the flow as symmetrical about the Oxz plane but this assumption will not be made because, although it *is* justifiable in level flight, when the aeroplane is banked the ailerons on the mainplane are brought into action and the flow will lose its symmetry. Furthermore, in the wind tunnel and aeroelastic applications of this theory there are no grounds for assuming symmetry.

3. General Discussion of Conical Flows

The remaining sections are concerned only with the tailplane and with the velocity potential ϕ'', defined in Section 2, so it will be convenient to drop the dashes and let ϕ be the velocity potential whose downwash w is given in the form

$$w=-(w_{00}+w_{10}x+w_{01}y+w_{20}x^2+w_{11}xy+w_{02}y^2+\ldots.). \quad . \quad . \quad (4)$$

Furthermore we shall call the aerofoil a wing, instead of a tailplane, and (4) gives the downwash on the wing. We also confine our attention to delta wings,

symmetrically placed with respect to the free stream direction, whose plan forms are isosceles triangles lying *inside* the Mach cone from the apex.

First order conical flows have been discussed in very great detail by a number of authors, in particular Multhopp[11], Germain[1, 2], Goldstein and Ward[3], Stewart[17], Lagerstrom[5], Ribner[13], and Lomax and Heaslet[7]. There is little point in repeating the majority of the first order theory but we will give sufficient to enable the reader to understand the higher order theory without reference to other papers.

We define a conical flow of the n^{th} order as a flow for which all the n^{th} partial derivatives of the velocity potential ϕ with respect to x, y or z are constant along straight lines through a fixed point. With this definition it is seen that the type of conical flow, considered by the authors cited, is of the first order, *i.e.* one in which the components of the perturbation velocity are all constant along straight lines through a fixed point, the vertex of the delta wing.

Higher order conical fields have been mentioned by Multhopp[11], and Lomax and Heaslet[7], and Germain[2] has also considered such flows but assumed that conditions were symmetrical about the plane $y=0$. This will not be sufficient for the present purpose and the theory is developed without any such assumptions. Multhopp only discussed first and second order theory although he did suggest how the work would proceed after that; his suggestion is however rather misleading.

4. Transformation of the Velocity Potential Equation—First Order Theory

The velocity components of the flow are given by $u=\partial\phi/\partial x$, $v=\partial\phi/\partial y$, and $w=\partial\phi/\partial z$, which all satisfy the Prandtl-Glauert equation. The equation for u is taken as typical.

$$\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=B^2\frac{\partial^2 u}{\partial x^2}. \qquad (5)$$

The co-ordinates (x, y, z) are changed to (R, ξ, θ) by means of the equations

$$x=BR\coth\xi, \quad y=R\operatorname{cosech}\xi\cos\theta, \quad z=R\operatorname{cosech}\xi\sin\theta. \qquad (6)$$

If a simple cone field is assumed, u is independent of R and it can be verified that (5) is transformed to

$$\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\theta^2}=0. \qquad (7)$$

One further transformation is now applied, namely,

$$\eta=\frac{\cos\theta\cosh\xi}{\cosh^2\xi-\sin^2\theta}, \qquad \zeta=\frac{\sin\theta\sinh\xi}{\cosh^2\xi-\sin^2\theta}. \qquad (8)$$

This change of variables does not change the form of (7), which becomes

$$\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 u}{\partial \zeta^2} = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

It can be shown that the transformations have reduced the problem to a two-dimensional one in the σ-plane. The $\sigma = \eta + i\zeta$ plane is such that the Mach cone has transformed into those parts of the real axis for which $-\infty \leqslant \eta \leqslant -1$ and $1 \leqslant \eta \leqslant \infty$ and the delta wing appears as that part of the real axis for which $|\eta| \leqslant a$, where $a = B/k$. (a is less than 1 for a wing with subsonic leading edges.)

We find that

$$\eta = \frac{Bxy}{(x^2 - B^2 z^2)} \quad . \quad . \quad . \quad . \quad . \quad (10)$$

and

$$\zeta = \frac{Bz\,(x^2 - B^2 y^2 - B^2 z^2)^{\frac{1}{2}}}{(x^2 - B^2 z^2)} \quad . \quad . \quad . \quad . \quad (11)$$

Since we are only interested here in the value of u and w *on* the delta wing, which is given by $|\eta| = |By/x| \leqslant a$, $\zeta = 0$, we are only concerned in the (η, ζ) plane with the values on the real axis. This fact makes for considerable simplification, since we need only evaluate the derivatives $\partial/\partial x$, $\partial/\partial y$ and $\partial/\partial z$ *on* the $\zeta = 0$ axis.

The results, which have been obtained before by Multhopp[11], are

$$\frac{\partial}{\partial x} = -\frac{\eta}{x}\frac{\partial}{\partial \eta}, \quad . \quad . \quad . \quad . \quad . \quad (12)$$

$$\frac{\partial}{\partial y} = \frac{B}{x}\frac{\partial}{\partial \eta} \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

and

$$\frac{\partial}{\partial z} = \frac{B(1-\eta^2)^{\frac{1}{2}}}{x}\frac{\partial}{\partial \zeta}, \quad . \quad . \quad . \quad . \quad (14)$$

all of which apply only when $\zeta = 0$ and $|\eta| < 1$. Remembering that the motion is irrotational, the following three equations connect u, v and w:—

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}. \quad . \quad . \quad (15)$$

Thus, using (12), (13) and (14) the irrotationality conditions become

$$\frac{\partial w}{\partial \eta} = (1-\eta^2)^{\frac{1}{2}}\frac{\partial v}{\partial \zeta}, \quad . \quad . \quad . \quad . \quad (16)$$

$$B(1-\eta^2)^{\frac{1}{2}}\frac{\partial u}{\partial \zeta} = -\eta\frac{\partial w}{\partial \eta} \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and

$$-\eta\frac{\partial v}{\partial \eta} = B\frac{\partial u}{\partial \eta}, \quad . \quad . \quad . \quad . \quad (18)$$

respectively, all of which are valid when y and $\eta > 0$.

5. The "Compatibility Conditions" in First Order Theory

Since, by definition, ϕ is a homogeneous function of x, y and z of degree one,

$$\phi = x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y} + z\frac{\partial\phi}{\partial z} = xu + yv + zw,$$

and therefore

$$x\,du + y\,dv + z\,dw = 0,$$

or

$$B\cosh\xi\,du + \cos\theta\,dv + \sin\theta\,dw = 0. \qquad (19)$$

From the fact that the velocity components satisfy Laplace's equation (7) inside the Mach cone it follows that $u(\theta, \xi)$, $v(\theta, \xi)$ and $w(\theta, \xi)$ are the real parts of analytic functions $U(\Theta)$, $V(\Theta)$ and $W(\Theta)$, where $\Theta = \theta + i\xi$.

Equation (19) can now be written

$$B\cosh\xi\,dU + \cos\theta\,dV + \sin\theta\,dW = 0,$$

and if $\overline{\Theta}$ denotes $\theta - i\xi$, the complex conjugate of Θ, it follows that

$$\cos\frac{\overline{\Theta}}{2}\left[B\cos\frac{\Theta}{2}\,dU + \cos\frac{\Theta}{2}\,dV + \sin\frac{\Theta}{2}\,dW\right] +$$

$$+ \sin\frac{\overline{\Theta}}{2}\left[B\sin\frac{\Theta}{2}\,dU - \sin\frac{\Theta}{2}\,dV + \cos\frac{\Theta}{2}\,dW\right] = 0.$$

Each bracket must vanish independently, and so

$$-B\,dU = \frac{dV}{\cos\Theta} = \frac{dW}{\sin\Theta}. \qquad (20)$$

These are the compatibility conditions and they connect the expressions for u, v and w. It will be convenient to apply the further transformation (8), namely

$$\sigma = \eta + i\zeta = \sec\Theta. \qquad (21)$$

In this case (20) becomes

$$-B\,dU = \sigma\,dV = -\frac{i\tau\,dW}{(1-\sigma^2)^{\frac{1}{2}}}. \qquad (22)$$

When considering the higher order conical fields, more and more analytic functions will be introduced and also more and more relationships connecting them. For instance in the present case, *i.e.* first order case, there are three functions, in the second there are six, in the third order there are ten and in the n^{th} order there are $\frac{1}{2}(n+1)(n+2)$ analytic functions. For this reason we introduce a notation which can be extended to n^{th} order flows. The equation (22) will be written

$$\frac{d\Phi_{100}}{d\sigma} = \frac{-\sigma}{B}\frac{d\Phi_{010}}{d\sigma} = \frac{i\sigma}{B(1-\sigma^2)^{\frac{1}{2}}}\frac{d\Phi_{001}}{d\sigma}, \qquad (23)$$

where the Φ's are analytic functions of σ and where we have replaced dU by $d\Phi_{100}$, dV by $d\Phi_{010}$ and dW by $d\Phi_{001}$. This notation can be extended to the general case (*see* equation (50)).

It is satisfying to note that the three relationships (16), (17) and (18) are in fact included in the compatibility conditions (22), bearing in mind that $\sigma=\eta+i\zeta$ and that $u=\mathscr{R}\,U$ and so on. The remaining equations of (22) merely give connections between the imaginary parts of U, V and W which are of no interest to us.

6. Expressions for Certain Derivatives on the $\zeta=0$ Axis

It is as well to pause here and summarise the problem. We wish to find the lift distribution on a symmetrical delta wing, which is swept inside the Mach cone, being given the downwash over the wing as a power series in x and y of the form (4). We are, therefore, interested in particular in the relations (22) which connect the downwash w with the lift distribution (which is a linear function of u). Examining (22) it is seen that $\partial/\partial\zeta$ appears in this equation, and since w is given in the $z=0\,(\zeta=0)$ plane we shall require an expression for $\partial(u,v,w)/\partial\zeta$ *on* $\zeta=0$; the next task is to obtain an expression for $\partial(u,v,w)/\partial\zeta$ on $\zeta=0$ in terms of the value of (u,v,w) on $\zeta=0$. Again u is taken as typical of the three velocity components.

For lifting problems

$$u(\eta,+0)=-u(\eta,-0)=u_1(\eta),\text{(say)}, \quad . \quad . \quad . \quad . \quad (24)$$

for $|\eta|<a$, where $\pm a$ are the extremities of the wing which appears as a cut along $\zeta=0$ in the σ-plane. Clearly,

$$u(\eta,\zeta)=\frac{\zeta}{\pi}\int_{-a}^{a}\frac{u_1(\eta')\,d\eta'}{(\eta-\eta')^2+\zeta^2} \quad . \quad . \quad . \quad . \quad . \quad (25)$$

gives the boundary values (24) along the wing cut and also it satisfies Laplace's equation (9). It now follows immediately that

$$\frac{\partial u(\eta,0)}{\partial\zeta}=\frac{1}{\pi}\int_{-a}^{a}\frac{u_1(\eta')\,d\eta'}{(\eta-\eta')^2} \quad \text{on} \quad \zeta=0. \quad . \quad . \quad . \quad (26)$$

This integral is improper if $|\eta|<a$ and the principal value of the integral is defined by

$$\frac{\partial u}{\partial\zeta}=\frac{1}{\pi}\lim_{\varepsilon\to 0}\left[\int_{-a}^{\eta-\varepsilon}\frac{u_1(\eta')\,d\eta'}{(\eta-\eta')^2}+\int_{\eta+\varepsilon}^{a}\frac{u_1(\eta')\,d\eta'}{(\eta-\eta')^2}-\frac{2u_1(\eta)}{\varepsilon}\right].$$

Mangler[9] has written an excellent report on improper integrals in which he proves that all that is necessary is to evaluate the integral in the ordinary straightforward way and any infinities which may occur can be rejected. A note on improper integrals also appears in the Appendix of Ref. 7. An integral of this sort, in which all infinities are to be neglected, is denoted here by the symbol $\oint$.

Subject to the condition that $u_1(-a)=u_1(+a)=0$, equation (26) may be written in the alternative form

$$\frac{\partial u}{\partial \zeta}=-\frac{1}{\pi}\oint_{-a}^{a}\frac{du_1}{d\eta'}\frac{d\eta'}{(\eta-\eta')}, \quad \text{on} \quad \zeta=0. \qquad (27)$$

A further relation, which can be proved by integrating by parts, and which will be used later is

$$\frac{d}{d\eta}\left[\oint_{-a}^{a}\frac{u_1(\eta')\,d\eta'}{(\eta-\eta')^n}\right]=\oint_{-a}^{a}\frac{du_1(\eta')}{d\eta'}\frac{d\eta'}{(\eta-\eta')^n}+\frac{u_1(-a)}{(\eta+a)^n}-\frac{u_1(a)}{(\eta-a)^n}. \qquad (28)$$

7. Higher Order Conical Flows

The theory considered so far will only enable us to solve the problem of finding the lift when $w=-w_{00}=$constant, on the wing; we now turn to the general problem and show how this may be dealt with. By definition, for a conical flow of order n the velocity potential $\phi(x, y, z)$ is a homogeneous function of degree n in (x, y, z). It follows that the n^{th} partial derivatives of ϕ with respect to x, y or z, or any combination of all three, are all homogeneous of order zero; in other words, if

$$\frac{\partial^n\phi}{\partial x^p\partial y^q\partial z^r}\equiv\phi_{pqr}, \qquad (29)$$

with $p+q+r=n$, then all the possible ϕ_{pqr} are functions of y/x, z/x and satisfy the two-dimensional Laplace equation (9) in the (η, ζ) plane. As in Section 5 it follows that ϕ_{pqr} may be taken as the real part of an analytic function of the complex variable σ, say, $\Phi_{pqr}(\sigma)$.

By Euler's Theorem on homogeneous functions it follows that

$$n!\,\phi(x, y, z)=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)^n\phi, \qquad (30)$$

where the bracket is to be expanded algebraically; the equation implies that if all the partial derivatives of ϕ, namely ϕ_{pqr}, are known, the ϕ itself is determined. Some particular partial derivatives are denoted by special symbols, for reasons which will be apparent later. Let

$$u_{nq}=\phi_{n-q,\,q,\,0} \quad \text{for all} \quad 0\leqslant q\leqslant n-1, \qquad (31)$$

also let

$$w_{nq}=\phi_{n-q-1,\,q,\,1} \quad \text{for all} \quad 0\leqslant q\leqslant n-1. \qquad (32)$$

If u and w are the x- and z-components of the velocity, respectively, then

$$(n-1)!\,u=\sum_{q=0}^{n-1}{}_qC_{n-1}u_{nq}x^{n-q-1}y^q \qquad (33)$$

and

$$(n-1)!\,w=\sum_{q=0}^{n-1}{}_qC_{n-1}w_{nq}x^{n-q-1}y^q \qquad (34)$$

} in the $z=0$ plane.

These definitions are consistent with the notation in the first order theory, because in the first order theory $n=1$, so $q=0$ only; thus $u_{10}=\phi_{100}=\partial\phi/\partial x$, so (33) shows that $u=u_{10}=\partial\phi/\partial x$, as it should; also $w_{10}=\phi_{001}=\partial\phi/\partial z$, so (34) gives $w=w_{10}=\partial\phi/\partial z$, again in agreement with the previous work.

8. The Compatibility Conditions in Higher Order Theory

The proof of formulae (22) is quite straightforward for first order conical flows, but it is not the best proof in extending the methods to second and higher order theories. For this reason an alternative derivation of (22) is given, incorporating the notation used in Section 7.

Beginning, as in Section 5, with the equation

$$x\,du+y\,dv+z\,dw=0, \quad . \quad . \quad . \quad . \quad . \quad (35)$$

let $\Phi_{100}(\sigma)=u+iu_1$, $\Phi_{010}=v+iv_1$ and $\Phi_{001}=w+iw_1$, where $\sigma=\sec\Theta$, as before. The component of the gradient of u_1 in any direction is equal to the component of the gradient of u in a direction at right angles; similarly for v and w; hence (35) implies $x\,du_1+y\,dv_1+z\,dw_1=0$, and so

$$x\Phi'_{100}(\sigma)+y\Phi'_{010}(\sigma)+z\Phi'_{001}(\sigma)=0, \quad . \quad . \quad . \quad . \quad (36)$$

the dash denoting differentiation with respect to σ. Differentiation of (36) with respect to x, y and z in turn leads to

$$\Phi'_{100}(\sigma)+[x\Phi''_{100}+y\Phi''_{010}+z\Phi''_{001}]\frac{\partial\sigma}{\partial x}=0 \quad . \quad . \quad . \quad (37)$$

and two similar equations. Thus

$$\Phi'_{100}:\Phi'_{010}:\Phi'_{001}=\frac{\partial\sigma}{\partial x}:\frac{\partial\sigma}{\partial y}:\frac{\partial\sigma}{\partial z}. \quad . \quad . \quad . \quad . \quad (38)$$

To find expressions for $\partial\sigma/\partial x$, $\partial\sigma/\partial y$ and $\partial\sigma/\partial z$, we pick out the coefficients of dx, dy and dz from the expression for $d\sigma$ in terms of x, y and z. Since $\sigma=\sec\Theta$ it can be shown that, from the transformations introduced by equations (6) and (8),

$$\frac{\partial\sigma}{\partial x}:\frac{\partial\sigma}{\partial y}:\frac{\partial\sigma}{\partial z}=-\frac{1}{B}:\frac{1}{\sigma}:\frac{i(1-\sigma^2)^{\frac{1}{2}}}{\sigma} \quad . \quad . \quad . \quad (40)$$

Thus (38) shows that

$$-B\frac{d\Phi_{100}}{d\sigma}=\sigma\frac{d\Phi_{010}}{d\sigma}=-\frac{i\sigma}{(1-\sigma^2)^{\frac{1}{2}}}\frac{d\Phi_{001}}{d\sigma} \quad . \quad . \quad . \quad (41)$$

which is the relation previously obtained (*see* equation (22)).

We now obtain the conditions when the flow is a second order conical flow. In this case ϕ is homogeneous of the second degree in (x, y, z). Hence

$$\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)^2\phi=2!\phi \quad . \quad . \quad . \quad . \quad (42)$$

and

$$x\frac{\partial\phi}{\partial x}+y\frac{\partial\phi}{\partial y}+z\frac{\partial\phi}{\partial z}=2\phi. \quad . \quad . \quad . \quad . \quad (43)$$

Therefore

$$x^2d\left(\frac{\partial^2\phi}{\partial x^2}\right)+2xyd\left(\frac{\partial^2\phi}{\partial x\partial y}\right)+y^2d\left(\frac{\partial^2\phi}{\partial y^2}\right)+z^2d\left(\frac{\partial^2\phi}{\partial z^2}\right)+2xzd\left(\frac{\partial^2\phi}{\partial x\partial z}\right)+2yzd\left(\frac{\partial^2\phi}{\partial y\partial z}\right)=0 \quad . \quad . \quad . \quad . \quad (44)$$

with all the second partial derivatives of ϕ with respect to x, y and z of degree zero. Each of these is thus the real part of an analytic function of σ. Letting

$$\frac{\partial^2\phi}{\partial x^2},\ \frac{\partial^2\phi}{\partial y^2},\ \frac{\partial^2\phi}{\partial z^2},\ \frac{\partial^2\phi}{\partial y\partial z},\ \frac{\partial^2\phi}{\partial x\partial z} \quad \text{and} \quad \frac{\partial^2\phi}{\partial x\partial y}$$

be the real parts of the six functions

$$\Phi_{200}(\sigma),\ \Phi_{020},\ \Phi_{002},\ \Phi_{011},\ \Phi_{101} \quad \text{and} \quad \Phi_{110}$$

respectively, (44) becomes

$$x^2\Phi'_{200}(\sigma)+2xy\Phi'_{110}+y^2\Phi'_{020}+2yz\Phi'_{011}+z^2\Phi'_{002}+2xz\Phi'_{101}=0. \quad . \quad (45)$$

Differentiating this equation twice, with respect to x, y and z in all possible ways gives the following six equations:—

$$\Phi'_{pqr}(\sigma)+\tfrac{1}{2}\left(x^2\Phi'''_{200}+2xy\Phi'''_{110}+y^2\Phi'''_{020}+2yz\Phi'''_{011}+z^2\Phi'''_{002}+2xz\Phi'''_{101}\right)\frac{\partial^2\sigma}{\partial x^p\partial y^q\partial z^r}=0, \quad . \quad (46)$$

$$p+q+r=2;$$

so that

$$\frac{d\Phi_{200}}{d\sigma}:\frac{d\Phi_{020}}{d\sigma}:\frac{d\Phi_{002}}{d\sigma}:\frac{d\Phi_{011}}{d\sigma}:\frac{d\Phi_{101}}{d\sigma}:\frac{d\Phi_{110}}{d\sigma}=$$

$$\frac{\partial^2\sigma}{\partial x^2}:\frac{\partial^2\sigma}{\partial y^2}:\frac{\partial^2\sigma}{\partial z^2}:\frac{\partial^2\sigma}{\partial y\partial z}:\frac{\partial^2\sigma}{\partial x\partial z}:\frac{\partial^2\sigma}{\partial x\partial y}. \quad (47)$$

By picking out the coefficients of d^2x and so on, from $d^2\sigma$, we can obtain $\partial^2\sigma/\partial x^2$ etc., and find that the resulting equations can be written compactly by saying that all the quantities

$$\frac{(-1)^p}{B^{p+q}}\sigma^p\left(\frac{i\sigma}{(1-\sigma^2)^{\frac{1}{2}}}\right)^q\frac{d\Phi_{2-p-q,\,p,\,q}(\sigma)}{d\sigma} \quad . \quad . \quad . \quad . \quad (48)$$

with $0 \leqslant p+q \leqslant 2$ are equal. It is easily seen that this expression can be generalised and, for n^{th} order conical flows, we have the very important and useful result that all the expressions

$$\frac{(-1)^p}{B^{p+q}} \sigma^p \left(\frac{i\sigma}{(1-\sigma^2)^{\frac{1}{2}}} \right)^q \frac{d\Phi_{n-p-q,\,p,\,q}(\sigma)}{d\sigma} \quad . \quad . \quad . \quad . \quad (49)$$

with $0 \leqslant p+q \leqslant n$ are equal.

We now have all the tools at our disposal and can proceed to solve the problem. First order theory deals with the case when ϕ is homogeneous of the first degree; so w, the downwash, is a constant on the wing. Second order theory deals with the case when w, on the wing, is proportional to x or y; third order theory when w, on the wing, is proportional to x^2, xy and y^2 and fourth order theory will give solutions to the case when w is proportional to x^3, x^2y, xy^2 and y^3, on the wing. The results of all these are given in Section 10; as the results indicate, the working is formidable even to third degree functions of x and y.

9. Method of Solution in the General Case

The working by which the results given in Section 10 are obtained will not be given for all cases but the third order case will be used to illustrate the procedure.

The third order theory will solve the cases in which the downwash is of the form

$$\begin{aligned} w &= -(w_{20}x^2 + w_{11}xy + w_{02}y^2) \\ &= \frac{1}{2!}(\phi_{201}x^2 + 2\phi_{111}xy + \phi_{021}y^2). \quad . \quad . \quad . \quad (50) \end{aligned}$$

The unknown function in this case can be taken to be $\partial^2 u/\partial x^2$, which is homogeneous of degree zero; this function is written ϕ_{300}. Thus to solve the case $w = -w_{20}x^2$, we have

$$\Phi_{201} = -2w_{20}, \quad \text{in} \quad |\eta| < a, \quad \text{on} \quad \zeta = 0 \quad . \quad . \quad (51)$$

and

$$\left.\begin{aligned} \Phi_{111} &= 0 \\ \Phi_{021} &= 0 \end{aligned}\right\} \quad \text{on} \quad \zeta = 0; \qquad \begin{aligned} &. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (52) \\ &. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (53) \end{aligned}$$

and we require to find Φ_{300} in $|\eta| < a$ on $\zeta = 0$. From (49) we pick out the three equations which connect Φ_{300} with Φ_{201}, Φ_{111} and Φ_{021}. They are

$$\frac{d\Phi_{300}}{d\sigma} = \frac{i\sigma}{B(1-\sigma^2)^{\frac{1}{2}}} \frac{d\Phi_{201}}{d\sigma} = \frac{-i\sigma^2}{B^2(1-\sigma^2)^{\frac{1}{2}}} \frac{d\Phi_{111}}{d\sigma} = \frac{i\sigma^3}{B^3(1-\sigma^2)^{\frac{1}{2}}} \frac{d\Phi_{021}}{d\sigma} \quad . \quad (54)$$

Equating real and imaginary parts of these equations, and equating the imaginary parts of Φ_{300}, Φ_{201}, Φ_{111} and Φ_{021} to zero (the real parts are ϕ_{300}, ϕ_{201}, ϕ_{111} and ϕ_{021}, respectively) we find

$$\frac{\partial \phi_{300}}{\partial \zeta} = \frac{-\eta}{B(1-\eta^2)^{\frac{1}{2}}} \frac{\partial \phi_{201}}{\partial \eta} = \frac{-\eta^3}{B^3(1-\eta^2)^{\frac{1}{2}}} \frac{\partial \phi_{021}}{\partial \eta} \quad . \quad . \quad . \quad (55)$$

The remaining equation is not required.

We see from (51) that

$$\frac{\partial \phi_{300}}{\partial \zeta} = 0 \quad \text{in} \quad |\eta| < a \quad \text{on} \quad \zeta = 0 \quad . \quad . \quad . \quad (56)$$

and from (55), since ϕ_{201} is zero at $\eta = 1$ and $-2w_{20}$ at $\eta = a$,

$$-B \oint_a^1 \frac{(1-\eta^2)^{\frac{1}{2}}}{\eta} \frac{\partial \phi_{300}}{\partial \zeta} d\eta = \oint_a^1 \frac{\partial \phi_{201}}{\partial \eta} d\eta = 2w_{20}, \quad \text{on} \quad \zeta = 0. \quad . \quad . \quad (57)$$

Using (27) it can be shown that

$$\frac{\partial \phi_{300}}{\partial \zeta} = \eta^2 \frac{\partial^2}{\partial \eta^2} \left\{ \frac{1}{\pi} \oint_{-a}^{a} \frac{u_1(\eta')\, d\eta'}{(\eta - \eta')^2} \right\}, \quad . \quad . \quad . \quad (58)$$

so by (56) we require the form of $u_1(\eta')$ such that

$$\frac{1}{\pi} \oint_{-a}^{a} \frac{u_1(\eta')\, d\eta'}{(\eta - \eta')^2} = \text{constant} + \eta \times \text{constant} \quad \text{in} \quad |\eta| < a.$$

A reference to the Appendix, result (*ii*), shows that this equation has a solution which is either symmetric or anti-symmetric. Only the symmetric solution is considered here, namely

$$Bu_1(\eta) = \frac{c_0 + c_2\eta^2}{(a^2 - \eta^2)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad . \quad (59)$$

where c_0 and c_2 are constants to be determined. From (58) and (59) we obtain

$$B \frac{\partial \phi_{300}}{\partial \zeta} = \eta^2 \frac{\partial^2}{\partial \eta^2} \left[\frac{(c_0 + 2c_2a^2)\eta - c_2\eta^3}{(\eta^2 - a^2)^{3/2}} \right]. \quad . \quad . \quad (60)$$

Substituting this value of $\partial \phi_{300}/\partial \zeta$ in (55) gives

$$\oint_a^1 \eta (1-\eta^2)^{\frac{1}{2}} \frac{\partial^2}{\partial \eta^2} \left[\frac{(c_0 + 2c_2a^2)\eta - c_2\eta^3}{(\eta^2 - a^2)^{3/2}} \right] d\eta = -2w_{20} \quad . \quad . \quad (61)$$

and

$$\oint_a^1 \frac{(1-\eta^2)^{\frac{1}{2}}}{\eta} \frac{\partial^2}{\partial \eta^2} \left[\frac{(c_0 + 2c_2a^2)\eta - c_2\eta^3}{(\eta^2 - a^2)^{3/2}} \right] d\eta = 0. \quad . \quad . \quad . \quad (62)$$

These integrals can be evaluated in terms of the complete elliptic integrals of the first and second kinds of modulus $(1-a^2)^{\frac{1}{2}}$. Equations (61) and (62) become

$$c_0[a^2(a^2-3)\,\mathrm{K}+2(2-a^2)\,\mathrm{E}]-c_2[2a^2\mathrm{K}-(1+a^2)\,\mathrm{E}]=2w_{20}(1-a^2)^2 \quad . \quad . \quad (63)$$

and $$c_0[2a^2(2a^2-1)\,\mathrm{K}+(4-7a^2+a^4)\,\mathrm{E}]+c_2[a^2(5a^2-3)\,\mathrm{K}+2(3-5a^2+a^4)\,\mathrm{E}]=0, \quad (64)$$

respectively. The solutions of (63) and (64) are

$$\left.\begin{aligned} c_0&=\frac{-2w_{20}a^2[a^2(3-5a^2)\,\mathrm{K}-2(3-5a^2+a^4)\,\mathrm{E}]}{[5a^4\mathrm{K}^2-8a^2(1+a^2)\,\mathrm{KE}-(4-19a^2+4a^4)\,\mathrm{E}^2]} \\ \text{and}\quad c_2&=\frac{+2w_{20}[2a^2(1-2a^2)\,\mathrm{K}-(4-7a^2+a^4)\,\mathrm{E}]}{[5a^4\mathrm{K}^2-8a^2(1+a^2)\,\mathrm{KE}-(4-19a^2+4a^4)\,\mathrm{E}^2]} \end{aligned}\right\} . \quad . \quad (65)$$

Since $u=x^2u_1(\eta)$ and the non-dimensional lift distribution $l(\eta)$ is given by

$$l(\eta)=\frac{4u}{U}=\frac{4x^2u_1(\eta)}{U}$$

the final result for the lift distribution is

$$l(x,y)=\frac{4k(c_0x^2+c_2B^2y^2)\,x}{UB^2(x^2-k^2y^2)^{\frac{1}{2}}}, \quad . \quad . \quad . \quad (66)$$

where c_0 and c_2 are given by (65).

In the following section the remaining results are listed, all working being omitted. It is realised that the working left out in the present section is considerable but it is given, in full, in Ref. 6.

10. List of Lift Distributions for Various Downwashes given on the Wing

(1) $w=-w_{00}$;

$$l(x,y)=\frac{4w_{00}x}{Uk\mathrm{E}(x^2-k^2y^2)^{\frac{1}{2}}}=\frac{4w_{00}x}{Uk\mathrm{E}X} . \quad . \quad . \quad . \quad (67)$$

(2) $w=-w_{10}x$;

$$l(x,y)=\frac{4w_{10}(1-a^2)(2x^2-k^2y^2)}{Uk[a^2\mathrm{K}+(1-2a^2)\,\mathrm{E}]X} . \quad . \quad . \quad . \quad (68)$$

(3) $w=-w_{01}y$;

$$l(x,y)=\frac{-4w_{01}(1-a^2)xy}{Uk[a^2\mathrm{K}-(2-a^2)\,\mathrm{E}]X} . \quad . \quad . \quad . \quad (69)$$

(4) $w=-w_{20}x^2$;

$$l(x,y)=\frac{4k(c_0x^2+c_2B^2y^2)\,x}{UB^2X}, \quad . \quad . \quad . \quad (70)$$

where

$$\left.\begin{aligned} c_0&=\frac{-2w_{20}a^2[a^2(3-5a^2)\,\mathrm{K}-2(3-5a^2+a^4)\,\mathrm{E}]}{[5a^4\mathrm{K}^2-8a^2(1+a^2)\,\mathrm{KE}-(4-19a^2+4a^4)\,\mathrm{E}^2]} \\ c_2&=\frac{+2w_{20}[2a^2(1-2a^2)\,\mathrm{K}-(4-7a^2+a^4)\,\mathrm{E}]}{[5a^4\mathrm{K}^2-8a^2(1+a^2)\,\mathrm{KE}-(4-19a^2+4a^4)\,\mathrm{E}^2]} \end{aligned}\right\} . \quad . \quad (71)$$

(5) $w = -w_{11}xy$;

$$l(x,y) = \frac{4ky\,(c_1x^2 + c_3B^2y^2)}{UBX}, \qquad (72)$$

where

$$\left.\begin{aligned} c_1 &= \frac{-2w_{11}a^2(1-a^2)^2}{B[a^2(1+a^2)\,\mathrm{K} - 2(1-a^2+a^4)\,\mathrm{E}]} \\ c_3 &= \frac{+w_{11}(1-a^2)^2}{B[a^2(1+a^2)\,\mathrm{K} - 2(1-a^2+a^4)\,\mathrm{E}]} \end{aligned}\right\} \qquad (73)$$

(6) $w = -w_{02}y^2$;

$$l(x,y) = \frac{4kx\,(c_4x^2 + c_6B^2y^2)}{UB^2X}, \qquad (74)$$

where

$$\left.\begin{aligned} c_4 &= \frac{+2w_{02}a^4[2a^2\mathrm{K} - (1+a^2)\,\mathrm{E}]}{B^2[5a^4\mathrm{K}^2 - 8a^2(1+a^2)\,\mathrm{KE} - (4-19a^2+4a^4)\,\mathrm{E}^2]} \\ c_6 &= \frac{-2w_{02}a^4[(3-a^2)\,\mathrm{K} - 2(2-a^2)\,\mathrm{E}]}{B^2[5a^4\mathrm{K}^2 - 8a^2(1+a^2)\,\mathrm{KE} - (4-19a^2+4a^4)\,\mathrm{E}^2]} \end{aligned}\right\}. \qquad (75)$$

(7) $w = -w_{30}x^3$;

$$l(x,y) = \frac{4k\,(c_8x^4 + c_{10}B^2x^2y^2 + c_{12}B^4y^4)}{UB^2X} \qquad (76)$$

where

$$\left.\begin{aligned} c_8 &= 8a^2(1-a^2)\,w_{30}[a^2(3-9a^2-2a^4)\,\mathrm{K} - (6-15a^2+5a^4-4a^6)\,\mathrm{E}]/D_1 \\ c_{10} &= -2(1-a^2)\,w_{30}[a^2(9-45a^2-4a^4)\,\mathrm{K} - (18-51a^2+a^4-8a^6)\,\mathrm{E}]/D_1 \\ c_{12} &= -2(1-a^2)\,w_{30}[a^2(9-a^2)\,\mathrm{K} - (3+7a^2-2a^4)\,\mathrm{E}]/D_1 \\ \text{and} & \\ D_1 &= a^4(27-31a^2-8a^4)\,\mathrm{K}^2 - 2a^2(24-5a^2-15a^4-16a^6)\,\mathrm{KE} - \\ &\quad -(12-119a^2+151a^4-64a^6+32a^8)\,\mathrm{E}^2. \end{aligned}\right\} \qquad (77)$$

(8) $w = -w_{21}x^2y$;

$$l(x,y) = \frac{4ky\,(c_5x^3 + c_7B^2xy^2)}{BUX}, \qquad (78)$$

where

$$\left.\begin{aligned} c_5 &= -2w_{21}a^2(1-a^2)[a^2(12-31a^2+27a^4)\,\mathrm{K} - (24-65a^2+55a^4-6a^6)\,\mathrm{E}]/BD_2 \\ c_7 &= +2w_{21}(1-a^2)[a^2(8-21a^2+21a^4)\,\mathrm{K} - (16-44a^2+39a^4-3a^6)\,\mathrm{E}]/BD_2 \\ \text{and} & \\ D_2 &= a^4(8+31a^2-27a^4)\,\mathrm{K} - 2a^2(16+15a^2+5a^4-24a^6)\,\mathrm{KE} + \\ &\quad +(32-64a^2+151a^4-119a^6+12a^8)\,\mathrm{E}^2 \end{aligned}\right\}. \qquad (79)$$

(9) $w=-w_{12}xy^2$;

Non-dimensional lift distribution given by (76) but in this case

$$\left.\begin{aligned} c_8&=8a^4(1-a^2)\,w_{12}\,[a^2(9-a^2)\,\mathrm{K}-(3+7a^2-2a^4)\,\mathrm{E}]/3B^2D_1\\ c_{10}&=-2a^2(1-a^2)\,w_{12}\,[a^2(81-65a^2+24a^4)\,\mathrm{K}+(3-131a^2+136a^4-\\ &\qquad -48a^6)\,\mathrm{E}]/3B^2D_1\\ c_{12}&=2(1-a^2)\,w_{12}\,[a^2(27-31a^2+12a^4)\,\mathrm{K}+\\ &\qquad +(6-55a^2+65a^4-24a^6)\,\mathrm{E}]/3B^2D_1 \end{aligned}\right\}. \quad (80)$$

and D_1 has the same value as before.

(10) $w=-w_{03}y^3$;

Lift distribution is given by (78) but in the present case

$$\left.\begin{aligned} c_5&=6w_{03}a^6\,(1-a^2)\,[a^2(1-9a^2)\,\mathrm{K}-(2-7a^2-3a^4)\,\mathrm{E}]/B^3D_2\\ c_7&=-6w_{03}a^6\,(1-a^2)\,[a^2(9-a^2)\,\mathrm{K}-(3+7a^2-2a^4)\,\mathrm{E}]/B^3D_2 \end{aligned}\right\}. \quad (81)$$

and D_2 has the same value as before. In all these results the modulus of the complete elliptic functions K and E is $(1-a^2)^{\frac{1}{2}}$.

The results (1) to (4), (6) (7) and (9) have been given before by various authors; the present results all agree with these authors. However the complexity of the results increases rapidly and it is desirable to be able to check them by an entirely independent method. This has been done by the present author using hyperboloidoconal co-ordinates; the method is too long to describe here but it is given in full in Ref. 6. The lift distributions so obtained do, in fact, completely confirm equations (67) to (81).

APPENDIX

SOLUTION OF CERTAIN INTEGRAL EQUATIONS

In Section 9 we had to solve integral equations which were of the form

$$\frac{1}{2\pi}\oint_{-a}^{a}\frac{g(\eta')\,d\eta'}{(\eta'-\eta)}=f(\eta)=\alpha+\beta\eta+\gamma\eta^2+\delta\eta^3. \quad . \quad . \quad . \quad (82)$$

Magnus[8] shows how to solve such an equation; we write $\eta=-a\cos\phi$ and $\eta'=-a\cos\psi$. If $F(\phi)=f(-a\cos\phi)$ may be expanded in a series of the form

$$\sin\phi\,F(\phi)=-\tfrac{1}{2}\sum_{n=1}^{\infty}b_n\sin n\phi,$$

then

$$g(\eta')=\left(1-\frac{\eta'^2}{a^2}\right)^{-\frac{1}{2}}(\tfrac{1}{2}b_0+\sum_{n=1}^{\infty}b_n\cos n\psi), \quad . \quad . \quad . \quad (83)$$

where b_0 is an arbitrary constant*. In the particular case when $f(\eta)$ has the form given by (82),

$$\begin{aligned} F(\phi) &= \alpha - \beta a \cos\phi + \gamma a^2 \cos^2\phi - \delta a^3 \cos^3\phi \\ &= [\alpha \sin\phi - \tfrac{1}{2}\beta a \sin 2\phi + \tfrac{1}{4}\gamma a^2 (\sin\phi + \sin 3\phi) - \tfrac{1}{8}\delta a^3 (2\sin 2\phi + \sin 4\phi)]/\sin\phi \\ &\equiv -\tfrac{1}{2}\sum_{n=1}^{\infty} b_n \frac{\sin n\phi}{\sin\phi}. \end{aligned}$$

Hence the constants b_r are

$$b_1 = -2(\alpha + \tfrac{1}{4}\gamma a^2), \quad b_2 = (\beta a + \tfrac{1}{2}\delta a^2), \quad b_3 = -\tfrac{1}{2}\gamma a^2, \quad b_4 = \tfrac{1}{4}\delta a^3$$

and $b_r = 0$ if $r > 4$. Thus, from (83),

$$\begin{aligned} g(\eta') &= a(a^2 - \eta'^2)^{-\frac{1}{2}} [\tfrac{1}{2} b_0 - (2\alpha + \tfrac{1}{2}\gamma a^2)\cos\psi + (\beta a + \tfrac{1}{2}\delta a^2)\cos 2\psi - \\ &\qquad - \tfrac{1}{2}\gamma a^2 \cos 3\psi + \tfrac{1}{4}\delta a^3 \cos 4\psi] \\ &= a(a^2 - \eta'^2)^{-\frac{1}{2}} [\tfrac{1}{2} b_0' - (2\alpha - \gamma a^2)\cos\psi + \\ &\qquad + (2\beta a - \delta a^2)\cos^2\psi - 2\gamma a^2 \cos^3\psi + 2\delta a^3 \cos^4\psi] \\ &\equiv (a^2 - \eta'^2)^{-\frac{1}{2}} (c_0 + c_1\eta' + c_2\eta'^2 + c_3\eta'^3 + c_4\eta'^4), \end{aligned}$$

where c_0 is an arbitrary constant and $c_1 = (2\alpha - \gamma a^2)$, $c_2 = (2\beta - \delta a)$, $c_3 = 2\gamma$ and $c_4 = 2\delta$.

Thus we have the following pairs of relationships:—

(*i*) When $f(\eta) = \alpha$,
$g(\eta') = (c_0 + c_1\eta')(a^2 - \eta'^2)^{-\frac{1}{2}}$.

(*ii*) When $f(\eta) = \alpha + \beta\eta$,
$g(\eta') = (c_0 + c_1\eta' + c_2\eta'^2)(a^2 - \eta'^2)^{-\frac{1}{2}}$.

(*iii*) When $f(\eta) = \alpha + \beta\eta + \gamma\eta^2$,
$g(\eta') = (c_0 + c_1\eta' + c_2\eta'^2 + c_3\eta'^3)(a^2 - \eta'^2)^{-\frac{1}{2}}$.

(*iv*) When $f(\eta) = \alpha + \beta\eta + \gamma\eta^2 + \delta\eta^3$,
$g(\eta') = (c_0 + c_1\eta' + c_2\eta'^2 + c_3\eta'^3 + c_4\eta'^4)(a^2 - \eta'^2)^{-\frac{1}{2}}$.

In all cases if the constants α, β, γ and δ are arbitrary, as they are in the present applications, then all the c_r's are also arbitrary. From these expressions, and using the symmetry or anti-symmetry of the required solutions, the solution of the integral equation used in Section 9 is as stated.

A further method of solving (82) is given by Sohngen[16]. Sohngen proves that if

$$-\frac{1}{2\pi} \oint_{-a}^{a} \frac{g(\eta')\, d\eta'}{(\eta' - \eta)} = f(\eta),$$

then

$$g(\eta') = c_0 (a^2 - \eta'^2)^{-\frac{1}{2}} - \frac{2}{\pi (a^2 - \eta'^2)^{\frac{1}{2}}} \oint_{-a}^{a} \frac{f(\eta)(a^2 - \eta^2)^{\frac{1}{2}}}{(\eta' - \eta)}\, d\eta. \quad . \quad . \quad (84)$$

*If $f(\eta)$ is discontinuous at $\eta = 0$, then (83) is not the complete solution. However, the $f(\eta)$'s considered in this paper are continuous at $\eta = 0$.

It can be verified that when $f(\eta)$ has the form given in (82) the expression for $g(\eta')$ given by (84) agrees with that obtained by using (83). When $f(\eta)$ has a form in which all the b_r's of (83) are non-zero then (84) is probably the simpler expression to use. Such forms of $f(\eta)$ have not however arisen in this paper.

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Some Examples of Three-Dimensional Effects in Boundary Layer Flow

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Summary: The three-dimensional boundary layer flow defined by the external irrotational velocity components $U=\chi(\xi)$, $V=\alpha\eta$, where α is a small positive parameter, is investigated with the aid of the boundary layer equations of Howarth[1]. When $\chi(\xi)=\xi^m$ a solution exact to the first power in α is found. A Pohlhausen method is then developed for any function $\chi(\xi)$ and applied to the cases in which $\chi(\xi)=\xi^m$ and $\chi(\xi)=1-\xi$.

Introduction

The boundary layer flow past an infinite yawed cylinder in a stream has already been extensively studied. Such a flow has external irrotational velocity components $U=\chi(\xi)$ perpendicular to the generators of the cylinder and $V=$constant parallel to the generators, and is one of the simplest cases of three-dimensional boundary layer flow. In order to obtain more information about three-dimensional effects in boundary layer flow, and especially on separation, consideration is given in this paper to another simple boundary layer flow: that defined by the external irrotational velocity components $U=\chi(\xi)$, $V=\alpha\eta$, where ξ, η are suitably chosen orthogonal curvilinear co-ordinates in the boundary surface and α is a small positive parameter. This flow, although chosen chiefly because its three-dimensional effects can be readily investigated, is not without practical interest, for it occurs in the flow of a stream past a body with a slightly curved leading edge, near the point at which the tangent to the edge is perpendicular to the stream.

In investigating this flow the three-dimensional boundary layer equations of Howarth[1] have been used and the solution expanded in powers of the parameter α, terms in α^2 and higher powers of α being neglected. If α is small enough this approximate solution will show adequately the trends of the three-dimensional effects on the flow. In general it is not possible to discuss a three-dimensional boundary layer flow without a knowledge of the geometry of the surface over which the flow takes place, for the curvature terms in Howarth's boundary layer equations depend on the shape of the surface. In this investigation these curvature terms are omitted, so that the equations apply strictly only to flow past a plane or a cylinder with velocities induced by external means. In practice, with many flows of the kind considered, the curvature terms are negligible, at least over a limited region.

In Part I a solution exact to the first power in α is obtained for various values of m when $\chi(\xi)=\xi^m$. In Part II a Pohlhausen method is developed for any function $\chi(\xi)$ and is then applied to the flow considered in Part I to give values for the skin

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friction in reasonable agreement with those obtained from the exact method. Finally the Pohlhausen method is applied to the case where $\chi(\xi)=1-\xi$. In this case it is known from two-dimensional applications that the Pohlhausen method gives poor correspondence with exact methods and the results obtained must be considered as qualitative.

In both cases, $\chi(\xi)=\xi^m$ and $\chi(\xi)=1-\xi$, the skin friction is found to be increased as the result of a cross flow of the type assumed. Because the method of solution breaks down for the critical value of m when $\chi(\xi)=\xi^m$ and the Pohlhausen solution gives increasingly inaccurate results with increase in ξ when $\chi(\xi)=1-\xi$, it is not possible to draw any certain conclusion about the separation of the flow: but in view of the increase in skin friction it seems reasonable to suppose that separation is delayed by the cross flow.

Notation

ξ, η, ζ	orthogonal curvilinear co-ordinates
u, v, w	velocity components inside the boundary layer
U, V	velocity components outside the boundary layer
μ	coefficient of viscosity
ν	kinematic viscosity
α	a small positive parameter
m	parameter in the expression $\chi(\xi)=\xi^m$
β	$=2m/(m+1)$
δ, δ'	boundary layer thicknesses
p	$=1/q=\delta/\delta'$
ρ	fluid density

PART I

$$U=\xi^m, \quad V=\alpha\eta.$$

The equations to be considered are

$$u\frac{\partial u}{\partial \xi}+v\frac{\partial u}{\partial \eta}+w\frac{\partial u}{\partial \zeta}=m\xi^{2m-1}+\nu\frac{\partial^2 u}{\partial \zeta^2}$$

$$u\frac{\partial v}{\partial \xi}+v\frac{\partial v}{\partial \eta}+w\frac{\partial v}{\partial \zeta}=\alpha^2\eta+\nu\frac{\partial^2 v}{\partial \zeta^2}$$

$$\frac{\partial u}{\partial \xi}+\frac{\partial v}{\partial \eta}+\frac{\partial w}{\partial \zeta}=0,$$

where $u \to \xi^m, \quad v \to \alpha\eta \quad \text{as} \quad \zeta \to +\infty$

$$u = v = 0 \quad \text{when} \quad \zeta = 0$$

and $u/\xi^m \to 1, \quad v/\alpha\eta \to 1 \quad \text{as} \quad \xi \to +0.$

These equations are derived from the boundary layer equations of Howarth[1] by taking $h_1 = h_2 = 1$ and substituting the pressure term derived from the irrotational flow $U = \xi^m$, $V = \alpha\eta$. The assumption that $h_1 = h_2 = 1$ means that the equations are strictly applicable only to flow over a plane or a cylinder with the given velocities induced by external means.

It is assumed that there is a solution $u = u(\xi, \zeta)$, $v = \alpha\eta v^*(\xi, \zeta)$, the assumption being justified *a posteriori*. In this case $w = w(\xi, \zeta)$. Expanding in powers of α

$$u = u_0(\xi, \zeta) + \alpha u_1(\xi, \zeta) + O(\alpha^2),$$
$$v^* = v_1(\xi, \zeta) + O(\alpha),$$
$$w = w_0(\xi, \zeta) + \alpha w_1(\xi, \zeta) + O(\alpha^2).$$

Substituting and equating coefficients of powers of α, the equations determining u_0, u_1, v_1 are

$$u_0 \frac{\partial u_0}{\partial \xi} + w_0 \frac{\partial u_0}{\partial \zeta} = m\xi^{2m-1} + \nu \frac{\partial^2 u_0}{\partial \zeta^2} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\frac{\partial u_0}{\partial \xi} + \frac{\partial w_0}{\partial \zeta} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$u_0 \frac{\partial v_1}{\partial \xi} + w_0 \frac{\partial v_1}{\partial \zeta} = \nu \frac{\partial^2 v_1}{\partial \zeta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$u_0 \frac{\partial u_1}{\partial \xi} + u_1 \frac{\partial u_0}{\partial \xi} + w_0 \frac{\partial u_1}{\partial \zeta} + w_1 \frac{\partial u_0}{\partial \zeta} = \nu \frac{\partial^2 u_1}{\partial \zeta^2} \quad . \quad . \quad . \quad . \quad (4)$$

$$\frac{\partial u_1}{\partial \xi} + v_1 + \frac{\partial w_1}{\partial \zeta} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

where $u_0 = u_1 = v_1 = 0 \quad \text{when} \quad \zeta = 0,$

and $u_0 \to \xi^m, \quad v_1 \to 1, \quad u_1 \to 0 \quad \text{as} \quad \zeta \to \infty.$

On writing $u_0 = \xi^m F_0'(y), \quad u_1 = \xi F_1'(y), \quad v_1 = G_1'(y),$

$$y = \left(\frac{m+1}{2\nu}\right)^{\frac{1}{2}} \xi^{\frac{m-1}{2}} \zeta, \quad \beta = \frac{2m}{m+1},$$

equations (1) and (2) reduce to

$$F_0'''+F_0F_0''=\beta\{(F_0)^2-1\}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

equation (3) becomes

$$G_1'''+F_0G_1''=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and equations (4) and (5) become

$$F_1'''+F_0F_1''-2F_0'F_1'+(3-2\beta)F_0''F_1=-(2-\beta)F_0''G_1, \quad . \quad . \quad (8)$$

where $F_0=F_0'=G_1=G_1'=F_1=F_1'=0$ when $y=0$

and $F_0'\longrightarrow 1, \quad G_1'\longrightarrow 1, \quad F_1'\longrightarrow 0$ as $y\longrightarrow\infty$.

Equation (6) is the Falkner-Skan equation in the form in which Hartree[2] used it. Hartree has given the solution for many values of β. Cooke[3] has considered equation (7), which can be solved by quadrature, and he gives the solution for some values of β.

The new equation, equation (8), has been solved for a few values of β and the quantities $F_1'(y)$, $G_1'(y)$, $F_1''(0)$, $G_1''(0)$ are shown in Table I.

$$\left(\frac{\partial u}{\partial\zeta}\right)_{\zeta=0}=\nu^{-\frac{1}{2}}\xi^{\frac{m-1}{2}}\left(\frac{m+1}{2}\right)^{\frac{1}{2}}\left\{\xi^m F_0''(0)+\alpha\xi F_1''(0)+O(\alpha^2)\right\}$$

$$\left(\frac{\partial v}{\partial\zeta}\right)_{\zeta=0}=\alpha\nu^{-\frac{1}{2}}\xi^{\frac{m-1}{2}}\eta\left(\frac{m+1}{2}\right)^{\frac{1}{2}}\left\{G_1''(0)+O(\alpha)\right\}$$

so that $F_1''(0)$ and $G_1''(0)$ are a measure of the change in skin friction caused by the three-dimensional motion.

The values of $F_1'(y)$, $G_1'(y)$, $F_1''(0)$, $G_1''(0)$, for $\beta=1$ have been taken from a paper by Howarth[4], who calculated them in finding the solution near a stagnation point. For $\beta=-0{\cdot}18$, equation (8) was solved by the Taylor series method and for the other values of β the central difference method was used.

When $\beta=0$ the solution is trivial, with

$$G_1(y)=F_0(y) \quad \text{and} \quad F_1(y)=\tfrac{1}{2}\{yF_0'(y)-F_0(y)\}$$

so that $G_1'(y)=F_0'(y)$ and $F_1'(y)=\tfrac{1}{2}yF_0''(y)$.

In this case $F_1''(0)=0{\cdot}2348$ and $G_1''(0)=0{\cdot}4696$.

When $\beta=2$ the solution of equation (8) is identically zero and when $\beta=-0{\cdot}1988$, which corresponds to two-dimensional separation, it seems to be impossible to find a solution which satisfies the boundary conditions. As $\beta\longrightarrow-0{\cdot}1988$ from above, $F_1''(0)\longrightarrow\infty$.

As α is positive, and supposed to be sufficiently small for terms in α^2 to be ignored, the skin friction is increased for each β, the increase becoming greater as

TABLE I

TABLE OF $F_1'(y)$ AND $G_1'(y)$

TABLE I

TABLE OF $F_1'(y)$ AND $G_1'(y)$

	$\beta=$	−0·18		+0·1		+0·2		+0·3		+1·0	
y		$F_1'(y)$	$G_1'(y)$	$F_1'(y)$	$G_1'(y)$	$F_1'(y)$	$G_1'(y)$	$F_1'(y)$	$G'(y)$	$F_1'(y)$	$G_1'(y)$
		0	0	0	0	0	0	0	0	0	0
0·2		0·153	0·076	0·037	0·098	0·030	0·101	0·026	0·104	0·010	0·114
0·4		0·306	0·152	0·073	0·196	0·060	0·202	0·051	0·207	0·019	0·228
0·6		0·458	0·228	0·108	0·293	0·088	0·302	0·074	0·310	0·027	0·339
0·8		0·607	0·304	0·139	0·388	0·113	0·400	0·095	0·409	0·033	0·446
1·0		0·750	0·379	0·166	0·480	0·133	0·494	0·110	0·505	0·036	0·547
1·2		0·884	0·452	0·185	0·566	0·147	0·582	0·120	0·594	0·037	0·639
1·4		1·002	0·523	0·196	0·647	0·153	0·663	0·123	0·675	0·035	0·720
1·6		1·098	0·591	0·198	0·719	0·151	0·735	0·111	0·747	0·031	0·789
1·8		1·165	0·656	0·190	0·782	0·142	0·797	0·097	0·808	0·026	0·846
2·0		1·198	0·716	0·175	0·836	0·128	0·849	0·082	0·859	0·021	0·891
2·2		1·192	0·770	0·154	0·880	0·110	0·891	0·066	0·899	0·016	0·926
2·4		1·146	0·818	0·130	0·915	0·090	0·924	0·051	0·930	0·012	0·951
2·6		1·065	0·860	0·104	0·942	0·071	0·949	0·037	0·954	0·008	0·969
2·8		0·953	0·895	0·081	0·961	0·053	0·967	0·026	0·970	0·005	0·981
3·0		0·822	0·924	0·060	0·975	0·038	0·979	0·017	0·981	0·003	0·988
3·2		0·681	0·946	0·042	0·985	0·026	0·987	0·011	0·989	0·002	0·993
3·4		0·541	0·963	0·029	0·991	0·017	0·993	0·007	0·994	0·001	0·996
3·6		0·413	0·976	0·019	0·995	0·011	0·996	0·004	0·996	0·001	0·998
3·8		0·302	0·984	0·012	0·997	0·006	0·998	0·002	0·998	0	0·999
4·0		0·212	0·990	0·007	0·998	0·004	0·999	0·001	0·999		1
4·2		0·143	0·994	0·004	0·999	0·002	0·9995	0	0·9995		
4·4		0·092	0·997	0·002	1	0·001	1		1		
4·6		0·056	0·998	0·001		0					
4·8		0·033	0·999	0·0005							
5·0		0·019	1	0							
5·2		0·010									
5·4		0·005									
5·6		0·002									
5·8		0·001									
6·0		0									

TABLE OF $F_1''(0)$ AND $G_1''(0)$

$F_1''(0)$	0·7658	0·1841	0·1516	0·1284	0·0478
$G_1''(0)$	0·3812	0·4909	0·5069	0·5195	0·5705

β decreases. The failure of the method of solution for $\beta=-0{\cdot}1988$ means that nothing definite can be said about the effect on separation, but the general increase in skin friction, together with the form of failure at $\beta=-0{\cdot}1988$, makes it reasonable to suppose that in fact separation will be delayed by a cross flow of this form.

One of the simplest applications of these results is to the flow near a stagnation point on a slightly curved, blunt leading edge—for example, to the flow near the forward stagnation point on a thin planetary ellipsoid of revolution placed in a stream parallel to an equatorial diameter. Then, with a suitable choice of co-ordinates, the external potential flow will be $U=\xi$, $V=\alpha\eta$—that is to say, in this case, $m=\beta=1$. Using the figures in Table I and those given by Hartree[2], it follows that the skin friction is $\rho\nu^{\frac{1}{2}}\xi\{1{\cdot}2326+0{\cdot}0478\,\alpha+O(\alpha^2)\}$.

The fractional increase, due to the introduction of the small cross flow V by the curvature of the leading edge, is to the first order in α, only $0{\cdot}0388\,\alpha$. In view of the fact that by hypothesis α is small, this change from the value for the two-dimensional flow past a straight edge is very small indeed.

PART II

With $h_1=h_2=1$ in the boundary layer equations, the appropriate momentum integral equations are

$$\frac{\partial}{\partial\xi}\int_0^\delta u^2d\zeta-U\frac{\partial}{\partial\xi}\int_0^\delta ud\zeta+\frac{\partial}{\partial\eta}\int_0^\delta uvd\zeta-U\frac{\partial}{\partial\eta}\int_0^\delta vd\zeta=\chi\chi'\delta-\nu\left(\frac{\partial u}{\partial\zeta}\right)_{\zeta=0} \quad (9)$$

$$\frac{\partial}{\partial\xi}\int_0^{\delta'} uvd\zeta-V\frac{\partial}{\partial\xi}\int_0^{\delta'} ud\zeta+\frac{\partial}{\partial\eta}\int_0^{\delta'} v^2d\zeta-V\frac{\partial}{\partial\eta}\int_0^{\delta'} vd\zeta=\alpha^2\eta-\nu\left(\frac{\partial v}{\partial\zeta}\right)_{\zeta=0} \quad (10)$$

where δ and δ' are the boundary layer thicknesses corresponding to the u and v velocities respectively.

Write $$u=\chi f(x,\lambda),\quad v=\alpha\eta g(y)$$

where
$$f(x,\lambda)\begin{cases}=(2x-2x^3+x^4)+\dfrac{\lambda x}{6}(1-x)^3 & \text{for } \zeta\leqslant\delta\\ =1 & \text{for } \zeta>\delta\end{cases}$$

$$g(y)\begin{cases}=(2y-2y^3+y^4) & \text{for } \zeta\leqslant\delta'\\ =1 & \text{for } \zeta>\delta'\end{cases}$$

$$x=\zeta/\delta,\quad y=\zeta/\delta',\quad \lambda=\chi'\delta^2/\nu.$$

If it is assumed that δ and δ' are functions of ξ but not of η—an assumption which can be justified *a posteriori* by finding such a solution satisfying the momentum integral equations—then, expanding in powers of α,

$$\delta=\varepsilon_0+\alpha\varepsilon_1+O(\alpha^2),\quad \delta'=\varepsilon_0'+O(\alpha),$$

where ε_0, ε_0', ε_1 are functions of ξ only. Substituting in equations (9) and (10) and equating coefficients of powers of α

$$\frac{d}{d\xi}\left\{\chi^2\varepsilon_0\int_0^1 f^2(x,\lambda_0)\,dx\right\}-\chi\frac{d}{d\xi}\left\{\chi\varepsilon_0\int_0^1 f(x,\lambda_0)\,dx\right\}=\chi\chi'\varepsilon_0-\frac{\chi\nu}{\varepsilon_0}\left(2+\frac{\lambda_0}{6}\right) \quad (11)$$

$$\frac{d}{d\xi}\left\{\chi q\varepsilon_0\int_0^1 f(qy,\lambda_0)\,[g(y)-1]\,dy\right\}=\frac{-2\nu}{q\varepsilon_0} \quad . \quad . \quad . \quad (12)$$

$$\frac{d}{d\xi}\left\{\chi^2\varepsilon_1\int_0^1 f^2(x,\lambda_0)\,dx+4\chi^2\varepsilon_1\int_0^1\lambda_0 f(x,\lambda_0)\frac{\partial f(x,\lambda_0)}{\partial\lambda_0}\,dx\right\}-$$

$$-\chi\frac{d}{d\xi}\left\{\chi\varepsilon_1\int_0^1 f(x,\lambda_0)\,dx+2\chi\varepsilon_1\int_0^1\lambda_0\frac{\partial f(x,\lambda_0)}{\partial\lambda_0}\,dx\right\}+$$

$$+\chi\varepsilon_0\int_0^1\left\{f(x,\lambda_0)-1\right\}g(px)\,dx=\chi\chi'\varepsilon_1+\frac{\chi\chi'\varepsilon_1}{\lambda_0}\left(2-\frac{\lambda_0}{6}\right) \quad . \quad (13)$$

where $p=1/q=\delta/\delta'$ and $\lambda_0=\chi'\varepsilon_0^2/\nu$.

Equation (11) is the ordinary Pohlhausen equation for the two-dimensional flow $U=\chi(\xi)$.

Equation (12) can be written

$$\frac{d}{d\xi}G(q)+G(q)\left\{\frac{1}{2}\left(\frac{1}{\lambda_0}\frac{d\lambda_0}{d\xi}-\frac{\chi''}{\chi'}\right)+\frac{\chi'}{\chi}\right\}=\frac{-2}{q\lambda_0}\frac{\chi'}{\chi} \quad . \quad . \quad (14)$$

where $G(q)=q\int_0^1 f(qy,\lambda_0)\,[g(y)-1]\,dy$

$$=\left(-\frac{7}{10}+\frac{13}{15}p-\frac{67}{140}p^3+\frac{7}{36}p^4\right)-\frac{(1-p)^4(3+2p)}{10p}+$$

$$+\lambda_0\left(-\frac{1}{120}+\frac{1}{180}p-\frac{1}{840}p^3+\frac{1}{3024}p^4\right) \quad \text{if} \quad p=1/q<1$$

$$=-q^2\left(\frac{2}{15}-\frac{3}{140}q^2+\frac{q^3}{180}\right)-\lambda_0 q^2\left(\frac{1}{90}-\frac{1}{84}q+\frac{3}{560}q^2-\frac{q^3}{1080}\right)$$

$$\text{if} \quad q<1.$$

On writing $\varepsilon_1=k(\xi)\,\varepsilon_0$, equation (13) becomes

$$\chi F_1(\lambda_0)\frac{dk}{d\xi}+\frac{k}{2}\left\{\left(\frac{1}{\lambda_0}\frac{d\lambda_0}{d\xi}-\frac{\chi''}{\chi}\right)\chi F_1(\lambda_0)+2\left(\chi\frac{dF_1(\lambda_0)}{d\xi}+\chi' F_2(\lambda_0)\right)\right\}-$$

$$-\frac{5\chi'}{3}-\frac{4\chi'}{\lambda_0}=-F_3(p,\lambda_0) \quad . \quad . \quad . \quad . \quad . \quad (15)$$

where
$$F_1(\lambda_0) = -\frac{37}{315} + \frac{\lambda_0}{315} + \frac{5\lambda_0^2}{9072}$$

$$F_2(\lambda_0) = \frac{293}{630} + \frac{79\lambda_0}{2520} + \frac{5\lambda_0^2}{4536}$$

$$F_3(p,\lambda_0)\begin{cases} = -p\left(\frac{2}{15} - \frac{3}{140}p^2 + \frac{1}{180}p^3\right) + \lambda_0 p\left(\frac{1}{180} - \frac{1}{840}p^2 + \frac{1}{3024}p^3\right) \text{ if } p<1. \\ = q\left(-\frac{7}{10} + \frac{13}{15}q - \frac{67}{140}q^3 + \frac{7}{36}q^4\right) - \frac{(3+2q)(1-q)^4}{10} + \\ \quad + \frac{\lambda_0}{6}\left[q^2\left(\frac{13}{30} - \frac{13}{14}q + \frac{201}{280}q^3 - \frac{7}{36}q^4\right) + \frac{(1+4q)(1-q)^4}{20}\right] \text{ if } p>1. \end{cases}$$

The method of solution for any given function $\chi(\xi)$ is as follows:—

(*i*) λ_0 is found from the equation

$$\frac{d\lambda_0}{d\xi} = \frac{\chi'}{\chi} g(\lambda_0) + \frac{\chi''}{\chi'}\left\{\lambda_0^2 h(\lambda_0) + \lambda_0\right\},$$

where $g(\lambda_0)$, $h(\lambda_0)$ are the functions defined in Ref. 5, page 160.

(*ii*) p or q is found from equation (14) which, when λ_0 is known, is a first order, first degree, non-linear differential equation. As $p, q \simeq 1$ usually, iteration is a suitable method of solution, beginning with $p(\xi)=1$.

(*iii*) k is found from equation (15) which, after the known values of p and λ_0 have been substituted, is a first order, linear differential equation.

It should be observed here that equation (14) is obtained when treating the flow past an infinite yawed cylinder by the Pohlhausen method: for the external velocity components are $U=\chi(\xi)$ and $V=$constant and the application of the Pohlhausen method leads at once to equations (11) and (12).

When p and k are known the velocities are found by expanding in powers of α:—

$$u = u_0 + \alpha u_1 + O(\alpha^2)$$

$$v = \alpha v_1 + O(\alpha^2)$$

$$u_0 = \chi(\xi)\left\{(2x - 2x^3 + x^4) + \frac{\lambda_0 x}{6}(1-x)^3\right\} \quad . \quad . \quad . \quad (16)$$

$$u_1 = -\chi(\xi)\, kx(1+2x)(1-x)^2\left(2 - \frac{\lambda_0}{6}\right) \quad . \quad . \quad . \quad (17)$$

where
$$x = \frac{\zeta}{\varepsilon_0} = \zeta\sqrt{\left\{\frac{\chi'(\xi)}{\lambda_0 \nu}\right\}}$$

$$v_1 = \eta(2y - 2y^3 + y^4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where $$y=\frac{p\zeta}{\varepsilon_0}=p\zeta\sqrt{\{\quad\}}.$$

In determining the skin friction it is observed that

$$\frac{\partial u}{\partial\zeta}=\frac{\partial u_0}{\partial\zeta}+\alpha\frac{\partial u_1}{\partial\zeta}+O(\alpha^2);\quad \frac{\partial v}{\partial\zeta}=\alpha\frac{\partial v_1}{\partial\zeta}+O(\alpha^2);$$

$$\left(\frac{\partial u_0}{\partial\zeta}\right)_{\zeta=0}=\chi(\xi)\sqrt{\left\{\frac{\chi'(\xi)}{\lambda_0\nu}\right\}}\left(2+\frac{\lambda_0}{6}\right)\ ;\quad \left(\frac{\partial u_1}{\partial\zeta}\right)_{\zeta=0}=-\chi(\xi)k\sqrt{\left\{\frac{\chi'(\xi)}{\lambda_0\nu}\right\}}\left(2-\frac{\lambda_0}{6}\right)$$

$$\left(\frac{\partial v_1}{\partial\zeta}\right)_{\zeta=0}=2p\eta\sqrt{\left\{\frac{\chi'(\xi)}{\lambda_0\nu}\right\}}.$$

The method has been applied to the flow considered in Part I, *i.e.* $\chi(\xi)=\xi^m$. In this case λ_0 is independent of ξ when m is given and Cooke[6] has shown that p is then also independent of ξ. He gives values of p for some values of $\beta(=2m/(m+1))$. When equation (15) is solved it is found that

$$k=C\xi^{-m+1}$$

where C depends only on β.

The quantities

$$\left(\frac{2}{m+1}\right)^{\frac{1}{2}}\nu^{\frac{1}{2}}\xi^{\frac{-(3m-1)}{2}}\left(\frac{\partial u_0}{\partial\zeta}\right)_{\zeta=0}=\left(2+\frac{\lambda_0}{6}\right)\sqrt{\left(\frac{\beta}{\lambda_0}\right)}=F_0''(\text{app.})\quad\text{say}$$

$$\left(\frac{2}{m+1}\right)^{\frac{1}{2}}\nu^{\frac{1}{2}}\xi^{\frac{-(m+1)}{2}}\left(\frac{\partial u_1}{\partial\zeta}\right)_{\zeta=0}=C\left(2-\frac{\lambda_0}{6}\right)\sqrt{\left(\frac{\beta}{\lambda_0}\right)}=F_1''(\text{app.})$$

$$\left(\frac{2}{m+1}\right)^{\frac{1}{2}}\nu^{\frac{1}{2}}\eta^{-1}\xi^{\frac{-(m-1)}{2}}\left(\frac{\partial v_1}{\partial\zeta}\right)_{\zeta=0}=2p\sqrt{\left(\frac{\beta}{\lambda_0}\right)}=G_1''(\text{app.})$$

are comparable with $F_0''(0)$, $F_1''(0)$, $G_1''(0)$ of Part I and are given in Table II together with $F_0''(0)$, $F_1''(0)$, $G_1''(0)$, λ_0 and $q(=1/p)$. When $\beta=0$, $\lambda_0=0$ and the equations as given cannot be used; but the method is simply modified in this case and Table II includes values for $\beta=0$.

It will be observed that the best value of $F_0''(\text{app.})$ occurs for a value of β between 0·1 and 0·2 and, as is well known, the agreement between $F_0''(\text{app.})$ and $F_0''(0)$ is good for all positive β. $F_1''(\text{app.})$ has its best value at some slightly negative value of β and the error increases as β increases. At $\beta=1$ it is 16·5 per cent. of $F_0''(0)$. The agreement between $G_1''(\text{app.})$ and $G_1''(0)$ is good for β positive and improves as β increases.

In each case as β becomes negative and tends to the value $-0{\cdot}1988$, the error increases rapidly.

TABLE II

β	−0·1988	−0·18	0	0·1	0·2	0·3	1·0	2·0
λ_0	−6·869	−5·426	0	1·434	2·509	3·371	7·052	11·984
q	0·857	0·878	1	1·048	1·089	1·126	1·304	1·340
F_0'' (app.)	0·146	0·200	0·485	0·591	0·683	0·764	1·196	1·638
F_0'' (0)	0	0·129	0·470	0·587	0·687	0·775	1·233	1·687
F_1'' (app.)	0·548	0·504	0·243	0·199	0·169	0·145	0·056	0
F_1'' (0)	—	0·766	0·235	0·184	0·152	0·128	0·048	0
G_1'' (app.)	0·397	0·417	0·485	0·504	0·519	0·530	0·577	0·610
G_1'' (0)	0·326	0·381	0·470	0·491	0·507	0·520	0·570	0·605

TABLE III

ξ	0	0·02	0·04	0·06	0·08	0·10	0·12	0·14	0·16
$-\lambda_0$	0	0·727	1·560	2·527	3·665	5·039	6·761	9·080	12·899
p	1	1·000	1·000	0·998	0·995	0·989	0·980	0·967	0·940
$-k$	0	0·010	0·022	0·034	0·048	0·066	0·088	0·12	0·3
$\frac{\nu^{\frac{1}{2}}}{1-\xi}\left(\frac{\partial u_1}{\partial \zeta}\right)_{\zeta=0}$	0	0·026	0·039	0·051	0·066	0·083	0·106	0·13	0·35
$\frac{\nu^{\frac{1}{2}}}{\eta}\left(\frac{\partial v_1}{\partial \zeta}\right)_{\zeta=0}$		2·346	1·601	1·256	1·039	0·881	0·754	0·64	0·52
$\left(\frac{\partial u_1}{\partial \zeta}\right)_{\zeta=0} \Big/ \left(\frac{\partial u_0}{\partial \zeta}\right)_{\zeta=0}$	0	0·012	0·028	0·052	0·091	0·161	0·315	0·83	
$\frac{(1-\xi)}{\eta}\left(\frac{\partial v_1}{\partial \zeta}\right)_{\zeta=0} \Big/ \left(\frac{\partial u_0}{\partial \zeta}\right)_{\zeta=0}$	1	1·065	1·149	1·264	1·432	1·705	2·245	3·97	

The other flow considered has $\chi(\xi)=1-\xi$. In this case the Pohlhausen method applied to the corresponding two-dimensional flow is known to give bad results and so a good approximation is not to be expected from the present method, which depends on the accuracy of the two-dimensional approximation: but the figures given in Table III should indicate the trend of the three-dimensional effects.

At the two-dimensional separation point, where $\lambda_0=-12$ and $\xi=0{\cdot}156$, $p=0{\cdot}949$, $k=0{\cdot}25$,

$$\frac{\nu^{\frac{1}{2}}}{1-\xi}\left(\frac{\partial u_1}{\partial\zeta}\right)_{\zeta=0}=0{\cdot}29,$$

$$\frac{\nu^{\frac{1}{2}}}{\eta}\left(\frac{\partial v_1}{\partial\zeta}\right)_{\zeta=0}=0{\cdot}55$$

approximately while the ratios

$$\left(\frac{\partial u_1}{\partial\zeta}\right)_{\zeta=0}\Big/\left(\frac{\partial u_0}{\partial\zeta}\right)_{\zeta=0} \text{ and } \frac{(1-\xi)}{\eta}\left(\frac{\partial v_1}{\partial\zeta}\right)_{\zeta=0}\Big/\left(\frac{\partial u_0}{\partial\zeta}\right)_{\zeta=0}$$

have become infinite.

It must be pointed out that the figures in Table III are not very accurate, especially for the larger values of ξ. More accurate values could be obtained only by using a smaller interval of ξ in solving the differential equations (14) and (15).

It can be shown, by using a method similar to that used in Part I, that there is an exact solution with

$$u_0=f_0'(z)+O(\xi)$$
$$u_1=\tfrac{1}{2}\xi z f_0''(z)+O(\xi^2)$$
$$v_1=\eta\{f_0'(z)+O(\xi)\}$$

where $z=\frac{1}{2}\zeta\xi^{-\frac{1}{2}}\nu^{-\frac{1}{2}}$ and u_0 is the Howarth solution[7] for the two-dimensional flow $U=1-\xi$.

From the expressions (16), (17) and (18) it can be shown that the Pohlhausen solution is of the form

$$u_0=F_0(z_1)+O(\xi)$$
$$u_1=\tfrac{1}{2}\xi z_1 F_0(z_1)+O(\xi^2)$$
$$v_1=\eta\{F_0(z_1)+O(\xi)\}$$

where $z_1=a\zeta\xi^{-\frac{1}{2}}\nu^{-\frac{1}{2}}$ and a is a certain constant ($=0{\cdot}1714$).

Thus
$$\left(\frac{\partial u_1}{\partial\zeta}\right)_{\zeta=0}\Big/\left(\frac{\partial u_0}{\partial\zeta}\right)_{\zeta=0}=\tfrac{1}{2}\xi+O(\xi^2)$$

and
$$\left(\frac{\partial v_1}{\partial\zeta}\right)_{\zeta=0}\Big/\left(\frac{\partial u_0}{\partial\zeta}\right)_{\zeta=0}=\eta\{1+O(\xi)\}.$$

The same expressions are obtained from the exact solution, so that it seems that near the origin of ξ the Pohlhausen method gives results in good agreement with the exact solution. It is impossible to determine at what value of ξ the

Pohlhausen and the exact solution begin to diverge seriously, but it is known that when $\xi=0{\cdot}120$ (which corresponds to two-dimensional separation in the exact solution) the Pohlhausen solution for the two-dimensional flow gives poor results: and, with the three-dimensional solution depending on the two-dimensional, the Pohlhausen solution must be in serious error for the larger values of ξ. However, it seems probable that the Pohlhausen solution will indicate the general trend of the three-dimensional effects and it will be seen from the figures in Table III that the skin friction is everywhere increased so that, although no certain conclusion can be drawn about the effect on separation, it seems reasonable to suppose that it will be delayed by the three-dimensional motion.

The figures do not indicate a large movement of the separation point as a consequence of the cross flow. To move the separation point from $\xi=0{\cdot}156$ to $\xi=0{\cdot}160$ would require a value of α of about $0{\cdot}12$.

Conclusion

In discussing the three-dimensional boundary layer flow defined by the external irrotational velocity components $U=\chi(\xi)$, $V=\alpha\eta$ it has been shown that in the particular cases for which $\chi(\xi)=\xi^m$ and $\chi(\xi)=1-\xi$ the skin friction is increased as the result of the cross flow, and although the breakdown of the method of solution for the critical value of m in one case, and the inadequacy of the solution for large value of ξ in the other, make it possible to draw only tentative conclusions about separation, it is likely that this will be delayed. Although it does not seem possible to give a complete physical explanation of these results at the present stage it is clear that there is a transfer of energy from the weaker cross flow to the stronger flow in the main stream direction which enables the flow to continue farther before the retarding action of the boundary causes separation.

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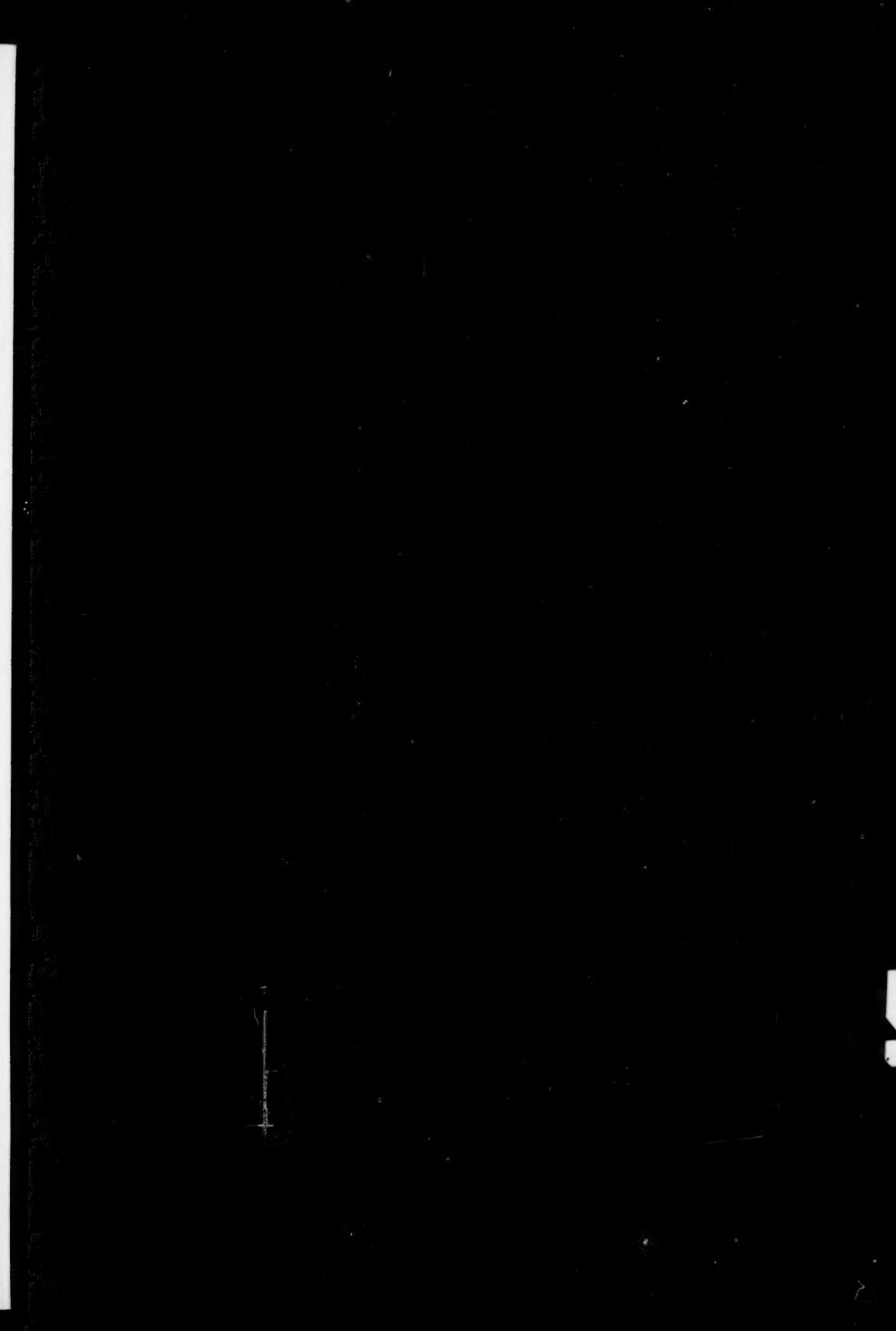
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